

# Estimating the Value at Risk of a bank's portfolio in sovereign bonds using a DCC-Copula model<sup>1</sup>

Jose E. Gomez-Gonzalez<sup>2</sup>  
Daniela X. Gualtero-Briceño<sup>3</sup>  
Luis F. Melo-Velandia<sup>4</sup>

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## Abstract

Rises in sovereign risk adversely affect banks reducing their profits and increasing their funding costs. Impacts are specially strong on banks holding important positions of government debt in the investment portfolios. This study applies a DCC-Copula model to estimate the  $VaR$  for a portfolio composed of 30 sovereign bonds from ten different countries and three different maturities. Results indicate that the model proposed in this study outperforms competing benchmark models under various back-testing criteria. The method here developed is useful for global banks holding a diversified portfolio of sovereign bonds, especially in emerging market countries in which banks mostly invest in public debt.

*Keywords:* Value at Risk, Banks' market risk, Dynamic copula models, Back-testing.

JEL Classification: C46, C52, C58, G32.

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<sup>2</sup>Corresponding author. Escuela Internacional de Ciencias Económicas y Administrativas, Universidad de La Sabana, Chía, Colombia. Email: jose.gomez2@unisabana.edu.co

<sup>3</sup>Banco de la República (Central Bank of Colombia), Financial Stability Department, Bogotá, Colombia. Email: dgualter@banrep.gov.co

<sup>4</sup>Banco de la República (Central Bank of Colombia), Econometrics Unit, Bogotá, Colombia. Email: lmelovel@banrep.gov.co

## 1. Introduction

The European sovereign debt crisis of 2009-2012 renewed interest in studying the link between country and bank risk. In fact, in late 2010 the Committee on the Global Financial System (CGFS) established a Study Group to examine the relationship between sovereign credit risk and bank funding conditions. Rises in sovereign risk adversely affect banks reducing their profits and increasing their funding costs. While banks' exposures are mostly to the home sovereign, large banks around the world hold also a diversified portfolio including other countries' sovereign bonds of diverse maturities. Albeit holding a diversified portfolio of sovereign bonds is beneficial from an asset diversification point of view, understanding market co-movement between these bonds and their impact on banks' market risk is crucial for enhancing financial stability.

Value at Risk ( $VaR$ ), the maximum loss that an investment portfolio will not exceed with a probability  $\alpha$  over a certain period of time, is an important tool for measuring and managing the effect of sovereign risk within a bank investment portfolio. A crucial reason for its popularity is that it is imposed by regulators to banks under the Basel II and Basel III accords. These accords suggest the  $VaR$  as the main approach for evaluating banks' market risk and as a major component for estimating the required banking capital.

Estimating the  $VaR$  of an investment portfolio is often challenging as this risk measure focus on extreme events for which there are usually few past observations to estimate statistical models. Parametric, semi-parametric, and non-parametric approaches are used for estimating the  $VaR$  of a given investment portfolio.

Parametric models are attractive as they are simple and work well for the mass of the distribution, but their performance in the tails is frequently disappointing especially during periods of financial turmoil. These models focus on estimating and forecasting the variance,  $\sigma^2$ , allowing for time variation due to volatility clusters. *RiskMetrics* and GARCH models are frequently used for this purpose.

$VaR$  estimation within an investment portfolio requires considering the dependence structure between asset returns. Copula functions provide a useful

framework for estimating the dependence structure. While more general than correlation coefficients, using copulas implies estimating a large number of parameters which increases exponentially with the number of assets included in the portfolio. This curse of dimensionality can be overcome by pairwise copula construction (see [17]). Multivariate GARCH models also suffer the same shortcoming which in empirical applications is commonly reduced by the implementation of the DCC-GARCH model.

Acknowledging that both copulas and the DCC-GARCH model offer compelling and complementary advantages in risk modeling, [22] propose the DCC-Copula model. This model has two main advantages over competing alternatives. First, it has relatively fewer parameters than other copula models. Second, the copula parameters associated with the second central moment of the series can vary over time. This is a great advantage of the model as correlations between asset returns usually present considerable time variation (see, for instance, [18]). For instance, common factors associated with the global business cycle ([9]) or with global uncertainty explain the high observed correlation between apparently unrelated financial assets and the considerable time variation of these correlations over time.

The DCC-Copula model has been used for modeling the dependence between different asset classes (e.g., [13]; [16]; [3]). A few number of recent applications to *VaR* estimation have also been implemented ([2]; [14]). This study contributes to the literature by estimating the *VaR* for a bank's investment portfolio composed of 30 sovereign bonds from ten different countries<sup>5</sup> using the DCC-Copula model. Estimation results, which are compared with those of other copula and multivariate GARCH models, indicate that the DCC-GARCH model provides the best performance among the benchmark models. These results indicate that the DCC-Copula model is an adequate approximation for measuring market risk and for calculating capital requirements in banking applications, especially for banks which are highly exposed to sovereign risk in their investment portfolios.

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<sup>5</sup>The US, The UK, Germany, Canada, France, Spain, Italy, Japan, Brazil and Colombia. Two large emerging market countries are included to account for assets in an emerging country's bank investment portfolio.

The remaining of the paper is presented in five sections. Section 2 contains a brief literature review. The third and fourth sections are methodological, describing the DCC-Copula model and the back-testing techniques used for evaluating the model's performance. Section 5 presents empirical results and the last section concludes.

## 2. Literature Review

Copula functions and multivariate GARCH models have been extensively used in financial risk applications. Due to advantages offered by these two approaches, recently the literature has focused on integrating them into a unified framework. [20] was the first in proposing a parametric method for estimating a copula function whose parameters vary over time. Specifically, in that seminal study the parameters of Symmetrized Joe-Clayton (SJC) and Gaussian copulas vary over time following an ARMA process. [15] apply dynamic GARCH-Copula function to model the dependence structure between four global financial indices. [1] use the same approach to model the conditional dependence structure between oil prices and exchange rates for a set of countries. [18] present a survey on the early literature on time-varying copulas.

The DCC-Copula model was introduced by [22]. The paper extends the DCC-GARCH model of [8] to allow for parameter variation over time using a Student's  $t$  copula. Specifically, the joint multivariate distribution is modeled using a time-varying copula. Recent studies have followed this approach in empirical financial applications. [13] focus on the dependence between US Treasury bond and other developed countries' asset returns. Their results show that the dependence structure between stock and 10-year government bond returns varies significantly over time for most countries. [16] apply the model to forecast financial market volatility using US stock market data. They show that the DCC-Copula model outperforms the linear time-varying regression model in volatility forecasting. [3] use the model to study the dynamic dependence between US stocks and commodities. They find stronger dependence relations in the long run, and report evidence of a leverage effect.

Less attention has been given to the implementation of the DCC-Copula approach to  $VaR$  estimations. [2] combines elliptical copulas with time varying

DCC matrices and Extreme Value Theory based models for the marginal return distributions for four different portfolios, two of them comprising stock market indices from five European countries, other comprising different German assets, and other containing exchange rates. Results indicate that the approach leads to reliable  $VaR$  estimations according to different back-testing criteria. [10] estimates the  $VaR$  for a portfolio composed of 30 Russian assets, while [11] presents an extension for high-dimensional diversified portfolios. [21] use various dynamic copula models for testing the  $VaR$  diversification hypothesis using US banks data. In a similar fashion, [14] also use the DCC-Copula and other dynamic copula models for estimating the  $VaR$  for a high-dimension portfolio composed of all companies listed in the CDX.NA.IG.

This paper contributes to the ongoing literature on  $VaR$  estimation using dynamic copula models by estimating the  $VaR$  for a bank's investment portfolio composed of 30 sovereign bonds from ten different countries. Three different bond maturities are included. We focus on government bonds as sovereign risk has become an issue for banks profitability and funding conditions along the last decade. Sovereign bonds constitute an important share of total investments for commercial and central banks around the world. Having an adequate measure of market risk is of major importance, as bank provisions and capital requirements depend on sound measures of this risk. This paper contributes to the development and implementation of better models for assessing the  $VaR$  and market risk for banks, especially those in emerging market economies in which the weight of public bonds on total banking investments is huge.

### 3. VaR estimation using the DCC-Copula method

The DCC-Copula approach combines the multivariate DCC-GARCH and Copula methods to model the behavior of the time-variant dependence of different series. Next, we describe each of these methods, beginning with the DCC-GARCH models.

#### 3.1. DCC-GARCH model

Let's define  $r_t$  as the vector containing the returns of  $N$  assets of a portfolio at time  $t$  with the following conditional distribution:

$$r_t | \Omega_{t-1} \sim (0, H_t), \quad t = 1, \dots, T \quad (1)$$

Where  $\Omega_t$  represents the history of the process up to time  $t$ ,  $H_t = D_t R_t D_t$ ,  $D_t$  is a diagonal matrix containing the time-conditional standard deviations at time  $t$  and  $R_t$  is the correlation matrix of the return series at time  $t$ .

The Dynamic Conditional Correlation model, DCC, proposed by [8], models the dynamics of conditional correlation,  $R_t$ . This model assumes that the conditional variance for the series follows a univariate *GARCH*( $p, q$ ) process. In other words:

$$\sigma_{i,t}^2 = \omega_i + \sum_{j=1}^p \gamma_{i,j} r_{i,t-j}^2 + \sum_{j=1}^q \lambda_{i,j} \sigma_{i,t-j}^2, \quad (2)$$

for  $i = 1, \dots, N$  and  $t = 1, \dots, T$ .

Afterwards, standardized errors  $\varepsilon_t$  are built as follows:

$$\varepsilon_t = D_t^{-1}(r_t - \mu_t) \quad (3)$$

$$\varepsilon_t \sim (0, R_t) \quad (4)$$

Equation (3) assume that the estimated mean of returns is constant; namely,  $\mu_t = \mu$ . More generally, it can be assumed that the first moment of the series follows an *ARMA*( $\tilde{p}, \tilde{q}$ ) process. In that case  $\mu_t$  is not constant and the return series  $r_{i,t}$  in equation (2) is replaced by the error term of the *ARMA* model.

Additionally, the DCC-GARCH model assumes that the conditional correlation matrices of standardized errors,  $\varepsilon_t$ , behave according the following

dynamics:<sup>6</sup>

$$Q_t = (1 - \alpha - \beta)\bar{Q} + \alpha\varepsilon_{t-1}\varepsilon'_{t-1} + \beta Q_{t-1} \quad (5)$$

$$\bar{Q} = \frac{1}{T} \sum_{t=1}^T \varepsilon_{t-1}\varepsilon'_{t-1} \quad (6)$$

Where  $\bar{Q}$  is the unconditional variance matrix of standardized errors.

Then, the correlation matrix,  $R_t$ , can be defined as:

$$R_t = (Q_t^*)^{-1/2} Q_t (Q_t^*)^{-1/2}$$

Where  $Q_t^* = \text{diag}(Q_t)$  and the elements of  $R_t$  have the following form:  $\rho_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{ii,t}q_{jj,t}}}$ ,  $i, j = 1, \dots, N$ ,  $t = 1, \dots, T$ .

According to equations (2) to (6), the DCC-GARCH model parameters can be classified into two groups. The first group contains the parameters of the univariate GARCH(p,q) models ( $\omega_i, \gamma_{i,j}, \lambda_{i,k}$ ,  $i = 1, \dots, N$ ,  $j = 1, \dots, p$ ,  $k = 1, \dots, q$ ). While the second group contains parameters that determine the dynamics of the correlation matrix  $R_t$  ( $\alpha$  and  $\beta$ ). Both sets of parameters can be estimated by maximizing the corresponding likelihood function.

In the context of the DCC-Copula model, the estimation of these parameters is slightly different from the methodology previously exposed. The DCC-Copula estimation will be explained later. But first, a brief explanation of copulas is presented in the next section.

### 3.2. Copula

A copula is a function used to model the dependence between a group of variables. Sklar's Theorem<sup>7</sup> shows that any multivariate distribution function of continuous random variables has an associated copula defined as a function of their marginal distributions:

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<sup>6</sup>These equations correspond to a DCC(1,1) model; however, they be generalized to a DCC(p,q) model.

<sup>7</sup>Described in [23]

**Teorema 1.** (*Sklar's theorem*). Let  $F(r_1, \dots, r_N)$  be the joint probability distribution function of  $N$  random variables  $r_1, \dots, r_N$ , with marginal distributions  $F_1(r_1), \dots, F_N(r_N)$ , respectively. Then, there exists a function  $C : [0, 1]^N \rightarrow [0, 1]$  such that:

$$F(r_1, \dots, r_N) = C(F_1(r_1), \dots, F_N(r_N)) \quad (7)$$

For example, the Gaussian copula has the following form:

$$C(u_1, u_2, \dots, u_N; R) = \Phi_R(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_N)), \quad (8)$$

where  $u_i$  corresponds to  $F_i(r_i)$  for  $i = 1, \dots, N$ ,  $\Phi(\cdot)$  denotes the univariate standard normal distribution function, and  $\Phi_R(\cdot)$  denotes the multivariate normal distribution function with correlation matrix  $R$ .

In practice, the joint distribution described in equation (7), is evaluated in the standardized errors related to the chosen returns. That is, over the errors presented in equation (3). Therefore:<sup>8</sup>

$$u_i = F_i(\varepsilon_i), \quad i = 1, \dots, N \quad (9)$$

### 3.3. DCC-Copula model

In the context of a DCC-Copula model, the copula parameters related to the correlation matrix vary overtime. For the example presented in (8),  $R_t$  is used instead of  $R$ . In this case, the dynamics of  $R_t$  is determined by a DCC model.

That is, for the DCC-Gaussian Copula model, we have the following equations:

$$C(u_1, u_2, \dots, u_N; R_t) = \Phi_{R_t}(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_N)), \quad (10)$$

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<sup>8</sup>The series  $u_i$  described in equation (9) can be estimated parametrically or non-parametrically, and the parameters associated with this transformation can also be estimated individually or jointly with the parameters of Copula. A detailed explanation of these methodologies can be found in [7], [4], [12] and [19], among others.

and the matrix  $R_t$  varies over time according to the DCC model:

$$\begin{aligned} R_t &= (Q_t^*)^{-1/2} Q_t (Q_t^*)^{-1/2}, \\ Q_t &= (1 - \alpha - \beta) \bar{Q} + \alpha \xi_{t-1} \xi'_{t-1} + \beta Q_{t-1}, \\ \bar{Q} &= \frac{1}{T} \sum_{t=1}^T \xi_{t-1} \xi'_{t-1}, \end{aligned} \quad (11)$$

where  $Q_t^* = \text{diag}(Q_t)$ ,  $\xi_t = (\xi_{1t}, \dots, \xi_{Nt})'$ ,  $\xi_{it} = \Phi^{-1}(u_{it})$ ,  $i = 1, \dots, N$ .

The parameters of the previous model are estimated by maximizing the following log-likelihood function:

$$l(R_t | u_1, \dots, u_N) = \sum_{t=1}^T \ln(c(u_{1t}, \dots, u_{Nt})), \quad (12)$$

where  $c(\cdot)$  represents the copula's density function. Meaning,  $c(u_1, \dots, u_N) = \frac{\partial^N C(u_1, \dots, u_N)}{\partial u_1, \dots, \partial u_N}$ .

In the case of a DCC-Gaussian Copula, the log-likelihood function is:

$$l(R_t | u_1, \dots, u_N) = -\frac{1}{2} \sum_{t=1}^T [ \log |R_t| + \xi'_t (R_t - I_N) \xi_t ] \quad (13)$$

where,  $R_t$  depends on  $\alpha$  and  $\beta$ , according to the expressions given in (11).<sup>9</sup>

### 3.4. VaR estimation

In the next section we will describe how to estimate the value at risk (VaR) using the DCC-Copula model. But first, we will introduce a brief definition of VaR.

The value at risk of an asset  $r$  at time  $t + 1$  with information up to time  $t$  is defined as the  $\alpha$ -quantile of its distribution function:

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<sup>9</sup>These results can be easily modified for the case of a DCC-t-Copula by changing expressions (10) and (13) with the proper equations of a t-Copula.

$$VaR_{t+1|t}^\alpha = F_{r(t+1|t)}^{-1}(\alpha) \quad (14)$$

Where  $F_r(\alpha)$  denotes the distribution function of  $r$ . For ease of interpretation let's define  $r$  as the negative returns of a particular asset. In this case, large values of  $r$  are associated with losses and not profits. Then, in terms of market risk events we are interested in the right tail of the distribution.<sup>10</sup>

For elliptical distributions the VaR can be calculated as follows:

$$VaR_{t+1|t}^\alpha = \mu_{t+1|t} + G_\alpha \sqrt{h_{t+1|t}}, \quad (15)$$

where  $\mu_{t+1|t}$  denotes the forecast of  $r$  at time  $t + 1$  with information up to  $t$ ,  $h_{t+1|t}$  is the variance forecast of  $r$  at time  $t + 1$  with information up to time  $t$  and  $G_\alpha$  is the  $\alpha$ -quantile of the profit and loss distribution.

VaR can also be calculated for a portfolio. A portfolio  $P_t$  is defined as,

$$P_t = \omega' r_t \quad (16)$$

where  $r_t = (r_{1t}, \dots, r_{Nt})'$  is a vector of  $N$  negative asset returns, and  $\omega = (\omega_1, \dots, \omega_N)'$ , is a vector of weights for the corresponding assets.

Then, assuming an elliptical distribution, the VaR of a portfolio  $P$  is calculated as follows:

$$VaR_{t+1|t}^\alpha = \omega' \mu_{t+1|t} + G_\alpha \sqrt{\omega' H_{t+1|t} \omega}, \quad (17)$$

where  $\mu_{t+1|t} = E(r_{t+1}|t)$ ,  $H_{t+1|t} = V(r_{t+1}|t)$  and  $G_\alpha$  is the  $\alpha$ -quantile of the distribution of the portfolio.

### 3.5. VaR estimation for DCC-Copula model

In the context of a DCC-Copula model, value at risk of a portfolio  $P$ , defined as in (16), can be estimated using simulations as follows:

1. Estimate parameters of the DCC-Copula model presented above. Specifically parameters of equations (2), (3), (9) and (11).

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<sup>10</sup>Constructing the we assume banks have a long position in bonds in their portfolio.

2. Forecast  $R_{t+1}$  with information up to time  $t$  according to equation (11).
3. Simulate  $M$  vectors of random numbers,  $r_{t+1}^m$ , with dimension  $N \times 1$  for  $m = 1, \dots, M$ . These must have the same marginal and joint distributions of the asset returns that are part of the portfolio. This simulation is carried out as follows:
  - Simulate  $N$  uniform random variables  $(0, 1)$ ,  $u_1, \dots, u_N$ , with the same dependence structure of the estimated copula. In particular, the correlation matrix of these  $N$  variables should be the one estimated in step 2,  $R_{t+1}$ .
  - Subsequently, standardized errors are calculated from the uniform variables simulated in the previous step, as follows:  $\varepsilon_{i,t+1} = F_i^{-1}(u_{i,t+1})$  for  $i = 1, \dots, N$ . Where  $F_i$  is the marginal distribution function of the standardized error of return  $i$ , whose parameters were estimated in step 1.
  - Then, the vector of returns is reconstructed for time  $t + 1$ ,  $r_{t+1} = (r_{1,t+1}, \dots, r_{N,t+1})'$  using equation (3). That is,  $r_{t+1} = D_{t+1}\varepsilon_{t+1} + \mu_{t+1}$ . The standardized errors are obtained from the previous step,  $\varepsilon_{t+1} = (\varepsilon_{1,t+1}, \dots, \varepsilon_{N,t+1})'$ .  $D_{t+1}$  and  $\mu_{t+1}$  are obtained as the forecasts of terms described in equation (3), which was estimated in step 1.
  - Finally, the returns of portfolio for time  $t + 1$  is easily calculated as  $\tilde{r}_{t+1} = \omega' r_{t+1}$ , where  $\omega$  is the vector of weights of the portfolio.
4. Once obtained the  $M$  simulations of the returns of the portfolio at time  $t + 1$ , calculate  $\widehat{VaR}_{t+1|t}^\alpha$  as the  $\alpha$ -quantile of the empirical distribution of these simulations.

#### 4. Backtesting

In concordance with the Basel committee on bank regulation, after calculating  $VaR$  is necessary to check that this is an appropriate measure of market risk.

First, let's assume that one-step ahead VaR for a confidence level  $\alpha$  is estimated from time  $n_1$  up to time  $n - 1$ :

$$VaR_{t+1|t}^\alpha, \text{ for } t = n_1, n_1 + 1, \dots, n - 1$$

Then, compute the exceptions function that compare the observed value of the returns with the estimated  $VaR$  is defined as follows:

$$I_{t+1|t}(\alpha) = \begin{cases} 1 & \text{si } r_{t+1} \geq VaR_{t+1|t}^\alpha \\ 0 & \text{si } r_{t+1} < VaR_{t+1|t}^\alpha \end{cases} \quad (18)$$

Under regular assumptions the exceptions series,  $I_{t+1|t}(\alpha)$ , has a Bernoulli ( $p$ ) distribution, where:

$$\begin{aligned} p &= 1 - \alpha \\ p &= E(I_{t+1|t}(\alpha)) \end{aligned}$$

According to [5], the backtesting procedures should evaluate that the exceptions series satisfies two properties.

1. **Unconditional Coverage Property (UC):** This property implies that the probability of occurrence of exceptions equal to one must be exactly  $1 - \alpha$ . If  $p > 1 - \alpha$ ,  $VaR$  will be underestimated, and if  $p < 1 - \alpha$ ,  $VaR$  will be overestimated.
2. **Independence property:** Under this property any pair of elements belonging to the exceptions series ( $I_{t+j|t+j-1}(\alpha)$ ,  $I_{t+k|t+k-1}(\alpha)$ ) must be independent. This guarantees that observed exceptions do not predict other future exceptions.

The Kupiec's proportion of failures is used to verify the unconditional coverage property, while the Christoffersen's tests are related with the unconditional coverage, the independence, and the conditional coverage properties. The last property is satisfied when the unconditional coverage and the independence properties are met.

### Kupiec's test

The null hypothesis of this test is  $H_0 : p = 1 - \alpha$  and can be evaluated using a likelihood ratio test with the following statistic:

$$LR_{CI} = -2 \ln \left( \frac{p^x (1-p)^{m-x}}{\hat{p}^x (1-\hat{p})^{m-x}} \right) = 2 \left( l(\hat{\Pi}_0) - l(\hat{\Pi}_p) \right) \quad (19)$$

Where  $x$  is the number of exceptions,  $m$  is the number of observations included in the backtesting and  $\tilde{p} = \frac{x}{m}$ . The numerator of the first term corresponds to the likelihood function under the null hypothesis and the denominator corresponds to the likelihood function evaluated in the not-restricted maximum likelihood estimator of  $p$ . The asymptotic distribution of this statistics under  $H_0$  is  $\chi^2$  with one degree of freedom.

### Christoffersen tests

[5] proposes to verify the appropriate number of exceptions and the independence property.

These tests are based on the assumption that the random variables  $I_{t+1|t}(\alpha)$  from  $n_1$  to  $n - 1$  follow a Markov chain of order one with the following transition matrix:

$$\Pi_1 = \begin{bmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{bmatrix} \quad (20)$$

Where  $\pi_{ij} = P(I_{t+1|t}(\alpha) = j | I_{t|t-1}(\alpha) = i)$  with  $i, j = \{0, 1\}$  are the transition probabilities. Then,  $\pi_{i0} + \pi_{i1} = 1$  for  $i = 0, 1$ .

Therefore, the transition matrix can be written as follows:

$$\Pi_1 = \begin{bmatrix} 1 - \pi_{01} & \pi_{01} \\ 1 - \pi_{11} & \pi_{11} \end{bmatrix} \quad (21)$$

Additionally, the likelihood function for  $I_{t+1|t}(\alpha)$  sequence is given by:

$$L(\Pi_1) = (1 - \pi_{01})^{m_{00}} \pi_{01}^{m_{01}} (1 - \pi_{11})^{m_{10}} \pi_{11}^{m_{11}} \quad (22)$$

Where  $m$  is the total number of observations of the backtesting procedure, and  $m_{ij}$  is the number of observations were  $I_{t+1|t}(\alpha) = j$  and  $I_{t|t-1}(\alpha) = i$  and consequently  $m_{00} + m_{01} + m_{10} + m_{11} = m$ .

Given (22), the maximum likelihood estimator of  $\Pi_1$  is:

$$\hat{\Pi}_1 = \begin{bmatrix} \hat{\pi}_{00} & \hat{\pi}_{01} \\ \hat{\pi}_{10} & \hat{\pi}_{11} \end{bmatrix} = \begin{bmatrix} \frac{m_{00}}{m_{00}+m_{01}} & \frac{m_{01}}{m_{00}+m_{01}} \\ \frac{m_{10}}{m_{10}+m_{11}} & \frac{m_{11}}{m_{10}+m_{11}} \end{bmatrix} \quad (23)$$

On the other hand, under the assumption of independence  $\pi_{01} = \pi_{11} = \pi$ . In this case, the transition matrix can be written as:

$$\Pi_0 = \begin{bmatrix} 1 - \pi & \pi \\ 1 - \pi & \pi \end{bmatrix} \quad (24)$$

Taking into account the restriction  $\pi_{01} = \pi_{11}$ , the maximum likelihood estimator of  $\pi$  is:

$$\hat{\pi} = \frac{m_{01} + m_{11}}{m} \quad (25)$$

Given that the null hypothesis associated to the independence property is  $H_0 : \pi_{01} = \pi_{11}$ , the likelihood ratio test statistic is:

$$LR_{Ind} = 2(l(\hat{\Pi}_1) - l(\hat{\Pi}_0)) \quad (26)$$

Where  $l(\cdot) = \log L(\cdot)$ . The asymptotic distribution of (26) under  $H_0$  is  $\chi^2$  with one degree of freedom. If  $H_0$  is rejected, the exceptions are not independent from each other.

In the case of the unconditional coverage property, [5] proposes the following likelihood ratio test statistic to evaluate the null hypothesis  $H_0 : p = 1 - \alpha$ :

$$LR_{UC} = 2(l(\hat{\Pi}_0) - l(\Pi_p)) \quad (27)$$

Where  $l(\cdot) = \log L(\cdot)$  and  $\Pi_p$  is the transition matrix evaluated at  $\pi = p = (1 - \alpha)$ . The asymptotic distribution of the statistics under  $H_0$  is  $\chi^2$  with one degree of freedom.

[5] also proposes to test the conditional coverage property. In this case, both the unconditional coverage and the independence are satisfied. The null

hypothesis is  $H_0 : \pi_{01} = \pi_{11} = p = 1 - \alpha$ , with the following test statistics:

$$LR_{CC} = 2(l(\hat{\Pi}_1) - l(\Pi_p)) \quad (28)$$

This statistics is asymptotically distributed  $\chi^2$  with 2 degrees of freedom under the null hypothesis.

## 5. Empirical Results

This paper computes the *VaR* for an equally weighted portfolio of 30 series. As the interest of a *VaR* relies on portfolio losses, we use the negative returns of zero coupon sovereign bonds of one, five and ten years to maturity for ten different countries, The US, The UK, Germany, Canada, France, Spain, Italy, Japan, Brazil, and Colombia. Bonds from the first eight countries are an important part of global banks' investment portfolios, while bonds from two large emerging market countries are included to account for the fact that banks in emerging market countries also hold an important portion of their investment portfolio in their own sovereign bonds. We use daily data from January 2010 to June 2019. The sample period covers the episode of the European sovereign crisis. Returns are calculated taking first differences of the zero coupon sovereign bonds' natural logarithms.

Table A.1 in Appendix A presents some descriptive statistics of the series of returns. It shows sample means, standard deviations, skewness, kurtosis, Jarque–Bera (JB) tests for normality, Ljung–Box tests for autocorrelation (LB), and the squared returns (LB2). Corresponding p-values are presented for the JB, LB, and LB2 tests. All series exhibit high kurtosis and most of them are also serially correlated, as it is often the case in financial time series. While under the null hypothesis of normal distribution excess kurtosis is equal to three, kurtosis values for all series in our sample are way higher. The JB test shows that returns are not normally distributed. Ljung–Box statistics for the squared series (LB2) show that a GARCH specification is adequate in this context due to the presence of volatility clusters in the data.

Our modelling strategy consists of four steps. In the first step, univariate AR-GARCH models are estimated for obtaining the standardized residuals for the 30 series in consideration. The second step consists in estimating the univariate marginal distribution of standardized residuals and applying the

probability integral transform to obtain uniformly distributed series. In the third step the parameters of a DCC-Copula model are estimated. Finally, the one-step ahead  $VaR$  of the equally weighted portfolio of the 30 series is estimated.

In the initial step univariate  $AR(p)$ -GARCH(1,1) models are estimated. Lag lengths  $p$  are chosen to minimize Akaike’s Information Criterion. Table B.2 in Appendix B shows test diagnosis for the standardized residuals of these models. Results indicate that there is no evidence of autocorrelation in the first and second moments.

Second, the univariate marginal distribution of the standardized residuals is estimated using the empirical distribution and applying the probability integral transform. Figure D.1 of Appendix D shows quantile-quantile plots of the previous series against a uniform reference distribution. The 45-degree line suggest that the uniform distribution provides a god fit for the data.

Third, the parameters of the DCC-Copula model are estimated, equations (10) and (11), using two elliptical copulas, the Gaussian copula and the t copula. These models are labeled DCC-Copula(t) and DCC-Copula(Gaussian), respectively. The results of the test of [6], that are presented in Table C.3 of Appendix C, indicate that the t Copula provides a better fit of the data.

Finally, the one-step ahead  $VaR$  of an equally weighted portfoliois estimated using bond return data from September 5, 2017 to June 3, 2019. This estimation is performed using the DCC-Copula model of the previous step using the simulation procedure described in section 3.5. For this purpose, 1000 simulations are used.

Results of the  $VaR$  for the DCC-Copula(t) and DCC-Copula (Gaussian) specifications are compared to those of four benchmark models, namely DCC(Gaussian), DCC(t), GARCH-Copula(Gaussian) and GARCH-Copula(t).<sup>11</sup> The DCC-Copula specifications, explained in section 3.3, model the second moment of the series using univariate GARCH models (equation (2)) and

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<sup>11</sup>In all specifications the first moment of the series is modeled as an  $AR(p)$  process. Lag lengths,  $p$ , are chosen by minimizing the AIC.

a dynamic joint dependence with a Copula (equations (10) and (11)). The DCC specifications model the second moment of the series using a multivariate GARCH model. Finally, the GARCH-Copula specifications model the second moment of the series through the implementation of univariate GARCH models and a static joint dependence structure with a copula function.

The DCC(Gaussian) and DCC(t) correspond to the DCC model described in equations (2) to (6), where the multivariate distribution of the standardized errors are the normal and t distributions, respectively. The models GARCH-Copula(Gaussian) and GARCH-Copula(t) assume that the return of the sovereign bonds follow a GARCH(1,1) Gaussian Copula and a GARCH(1,1) t-Copula, respectively. In both cases the marginal distributions of the standardized errors are estimated with an empirical distribution.

To evaluate the performance of these methodologies, the one-step ahead *VaR* is computed for the 6 models for the 95, 99, and 99.5 percentiles. For this purpose, the *VaR* is initially estimated using the information available up to September 5, 2017. Thereafter, the information set is augmented by one observation at a time, and the *VaR* is estimated again. This procedure is implemented recursively 450 times, until the information set reaches the period June 3, 2019.

Figures F.2, F.3, F.4, F.5, F.6 and F.7 of Appendix F correspond to the back-testing graphs for the six models. The figures are presented in three panels, one for each of the percentiles of the *VaR*. The solid blue line represents the portfolio return, the dotted red line is the one-step ahead *VaR*, and the solid black circles indicate the exceptions, i.e., when the *VaR* is greater than the return series. As expected, there are few exceptions and the *VaR* resembles the volatility of the series.

Table E.4 of Appendix E show the [5] back-testing test results for the 95 percentile *VaR* for the six models. Similarly, Tables E.5 and E.6 present the results for percentiles 99 and 99.5, respectively.

The rows of these tables indicate the six models that were used to estimate the portfolio *VaR*: DCC(Gaussian), DCC(t), GARCH-Copula(Gaussian), GARCH-Copula(t), DCC-Copula(Gaussian) and DCC-Copula(t). The columns

are associated with the Christoffersen back-testing tests (independence, unconditional coverage, and conditional coverage). For percentiles 99 and 99.5, the results presented in Tables E.5 and E.6 indicate that the null hypotheses of the three tests are not rejected for most models.

However, the results of Table E.4 for percentile 95 show that the model DCC-Copula(t) is the only for which the Christoffersen's tests are not rejected at the 5% significance level. This means that the DCC-Copula(t) model outperforms the other five models in terms of the back-testing results.

## 6. Conclusions

Sovereign risk affects bank stability impacting banks' profitability and their funding conditions. While banks' exposures are mostly to the home sovereign, large banks around the world hold also a diversified portfolio including other countries' sovereign bonds of diverse maturities. Therefore, analyzing risk exposure to a portfolio composed by government bonds from various countries and maturities is important for assessing market risk exposure of global banks.

In this paper we estimate the  $VaR$  of a diversified portfolio composed by 30 sovereign bonds of ten countries using a DCC-Copula model that combines the multivariate DCC-GARCH approach with a dynamic Copula specification. Given that there is no close solution, we estimate the  $VaR$  for this model using a simulation procedure. Performance of the model is compared with those of other benchmark models.

Results indicate that the DCC-Copula model proposed in this study outperforms competing benchmark models under various back-testing criteria. Hence, modeling the second moment of a series of returns requires accounting both for the dependence structure of returns and for their dynamic behavior over time.

The method for computing the  $VaR$  in this study is useful for global banks holding a diversified portfolio of sovereign bonds, especially in emerging market countries in which banks mostly invest in public debt.

## Appendix A. Descriptive Statistics

Table A.1: Descriptive statistics

	Mean	SD	Skewness	Kurtosis	JB	LB	LB2
COLOMBIA-1Y	-0.0001	0.052	-0.368	6.928	0	0.0004	0
COLOMBIA-5Y	0.001	0.063	-0.530	8.968	0	0.005	0
COLOMBIA-10Y	0.001	0.067	-0.099	7.726	0	0.070	0
BRAZIL-1Y	0.002	0.075	-0.100	7.603	0	0.0000	0
BRAZIL-5Y	0.003	0.125	-0.023	6.226	0	0.124	0
BRAZIL-10Y	0.002	0.142	0.025	6.812	0	0.188	0
USA-1Y	-0.001	0.014	0.506	8.697	0	0	0
USA-5Y	0.0004	0.046	0.056	4.464	0	0.021	0
USA-10Y	0.001	0.050	-0.119	4.205	0	0.003	0
CANADA-1Y	-0.001	0.020	0.268	7.780	0	0.077	0
CANADA-5Y	0.001	0.041	0.115	3.848	0	0.063	0
CANADA-10Y	0.001	0.042	-0.074	3.391	0	0.222	0
GERMANY-1Y	0.001	0.021	0.099	12.048	0	0	0
GERMANY-5Y	0.001	0.037	0.010	5.011	0	0	0
GERMANY-10Y	0.001	0.042	-0.220	4.666	0	0.002	0
UK-1Y	-0.00001	0.022	0.014	4.776	0	0.039	0
UK-5Y	0.001	0.042	0.006	3.645	0	0.496	0
UK-10Y	0.001	0.048	-0.009	3.633	0	0.039	0
FRANCE-1Y	0.001	0.018	0.624	13.276	0	0	0
FRANCE-5Y	0.001	0.038	0.066	7.014	0	0	0
FRANCE-10Y	0.001	0.041	-0.225	4.872	0	0	0
SPAIN-1Y	0.001	0.075	-0.308	24.234	0	0	0
SPAIN-5Y	0.001	0.078	0.591	11.547	0	0	0
SPAIN-10Y	0.001	0.072	0.265	8.726	0	0	0
ITALY-1Y	0.001	0.082	-0.203	21.358	0	0	0
ITALY-5Y	0.001	0.088	0.541	13.380	0	0	0
ITALY-10Y	0.0005	0.072	-0.229	7.177	0	0	0
JAPAN-1Y	0.0002	0.008	0.669	13.890	0	0	0
JAPAN-5Y	0.0003	0.013	-0.489	8.384	0	0.014	0
JAPAN-10Y	0.001	0.018	-0.356	6.664	0	0.007	0

JB represents the Jarque–Bera test. LB stands for Ljung–Box test over the returns, while LB2 does it for Ljung–Box test on the squared returns. For all three tests the p-values are presented.

## Appendix B. Residual diagnostics

Table B.2: Multivariate specification tests for the standardized residuals

	lags	Statistic	P-Value
Portmanteau	250	224786.5	0.62
Portmanteau (squared residuals)	250	216413.8	0.99

Ljung-Box test for the standardized residuals and the squared standardized residuals. The null hypotheses indicates indicates that there is no autocorrelation.

## Appendix C. Copula Selection

Table C.3: Copula Selection

Klic	P-Value
437.08	$< 0.001$

Test of [6]. The null hypotheses indicates a there is no difference between a Gaussian and a t Copula, while the alternative indicates a t Copula.

## Appendix D. QQ-plot of pseudo-sample

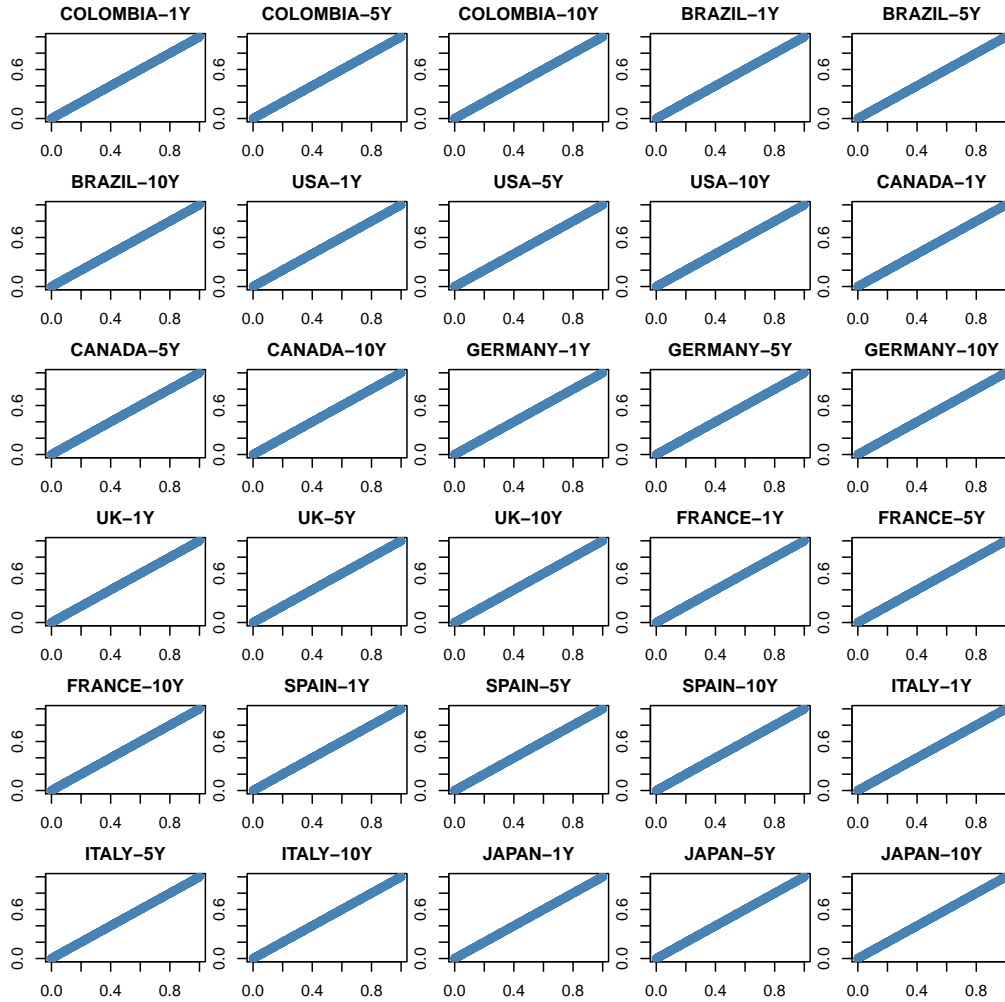


Figure D.1: QQ-plot of  $\hat{F}(\hat{\varepsilon}_i)$  against a uniform reference distribution.  $\hat{\varepsilon}_i$  corresponds to the standardized residuals for each of the 30 series that are considered and  $\hat{F}(x)$  is the empirical distribution of  $x$ .

## Appendix E. Backtesting Results

Table E.4: P-Values of Christoffersen tests for  $VaR(0.95)$

Model	LR.uc	LR.ind	LR.cc
DCC(Gaussian)	0.005	0.258	0.011
DCC(t)	0.024	0.376	0.052
GARCH-Copula(Gaussian)	0.024	0.048	0.011
GARCH-Copula(t)	0.002	0.207	0.004
DCC-Copula(Gaussian)	0.024	0.048	0.011
DCC-Copula(t)	0.078	0.089	0.050

*LR.uc* stands for the unconditional coverage test, *LR.ind* for independence test and *LR.cc* for the conditional coverage test.

Table E.5: P-Values of Christoffersen tests for  $VaR(0.99)$

Model	LR.uc	LR.ind	LR.cc
DCC(Gauss)	0.178	0.894	0.400
DCC(t)	0.178	0.894	0.400
GARCH-Copula(Gaussian)	0.178	0.894	0.400
GARCH-Copula(t)	0.043	0.947	0.130
DCC-Copula(Gaussian)	0.178	0.894	0.400
DCC-Copula(t)	0.178	0.894	0.400

*LR.uc* stands for the unconditional coverage test, *LR.ind* for independence test and *LR.cc* for the conditional coverage test.

Table E.6: P-Values of Christoffersen tests for  $VaR(0.995)$

Model	LR.uc	LR.ind	LR.cc
DCC(Gaussian)	0.854	0.894	0.975
DCC(t)	0.342	0.947	0.635
GARCH-Copula(Gaussian)	0.342	0.947	0.635
GARCH-Copula(t)	0.033	0.999	0.103
DCC-Copula(Gaussian)	0.342	0.947	0.635
DCC-Copula(t)	0.342	0.947	0.635

*LR.uc* stands for the unconditional coverage test, *LR.ind* for independence test and *LR.cc* for the conditional coverage test.

## Appendix F. Backtesting Graphics

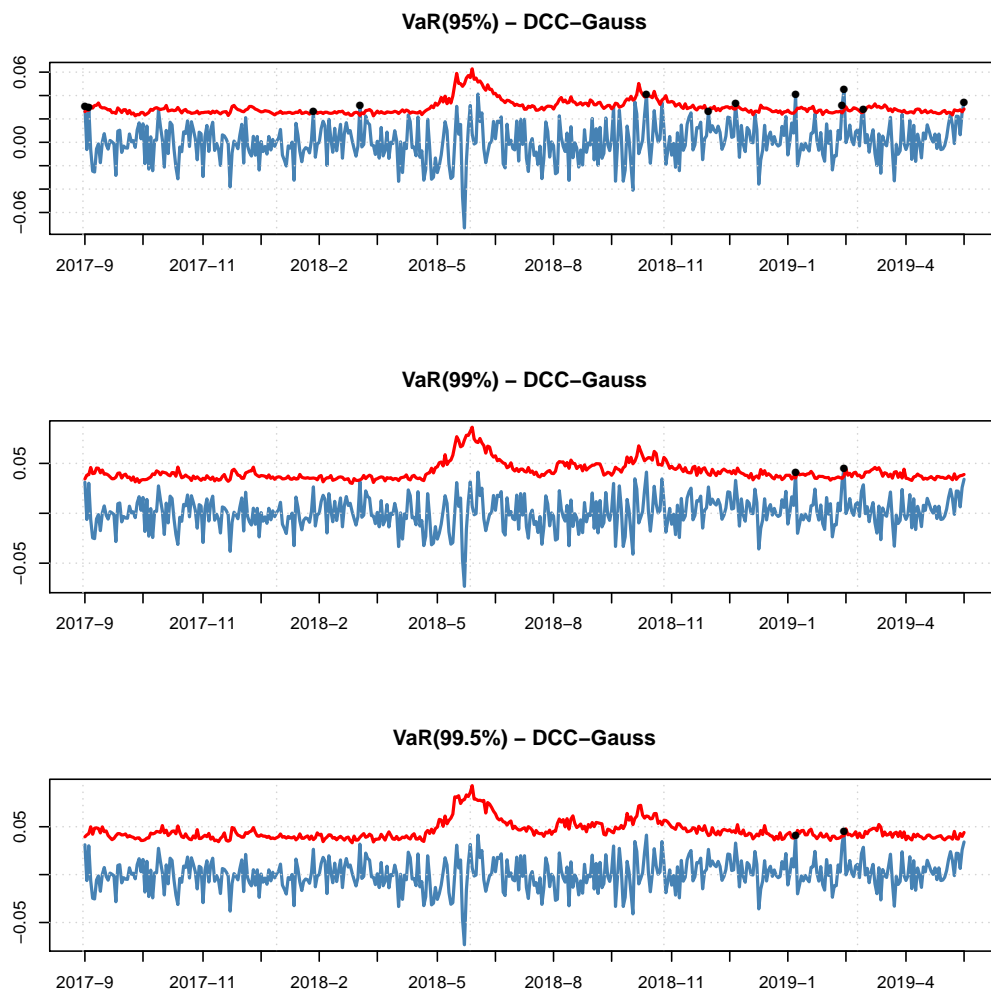


Figure F.2: Backtesting graphs for one-day ahead VaR of an equally weighted sovereign bond portfolio of ten countries and three maturities using a DCC model with normal distribution. The upper panel shows the VaR with a 0.05 significance level, 0.01 for the middle panel and 0.005 for the bottom one. The solid line is the portfolio returns, the dotted line corresponds to the VaR and the solid circles are the exceptions.

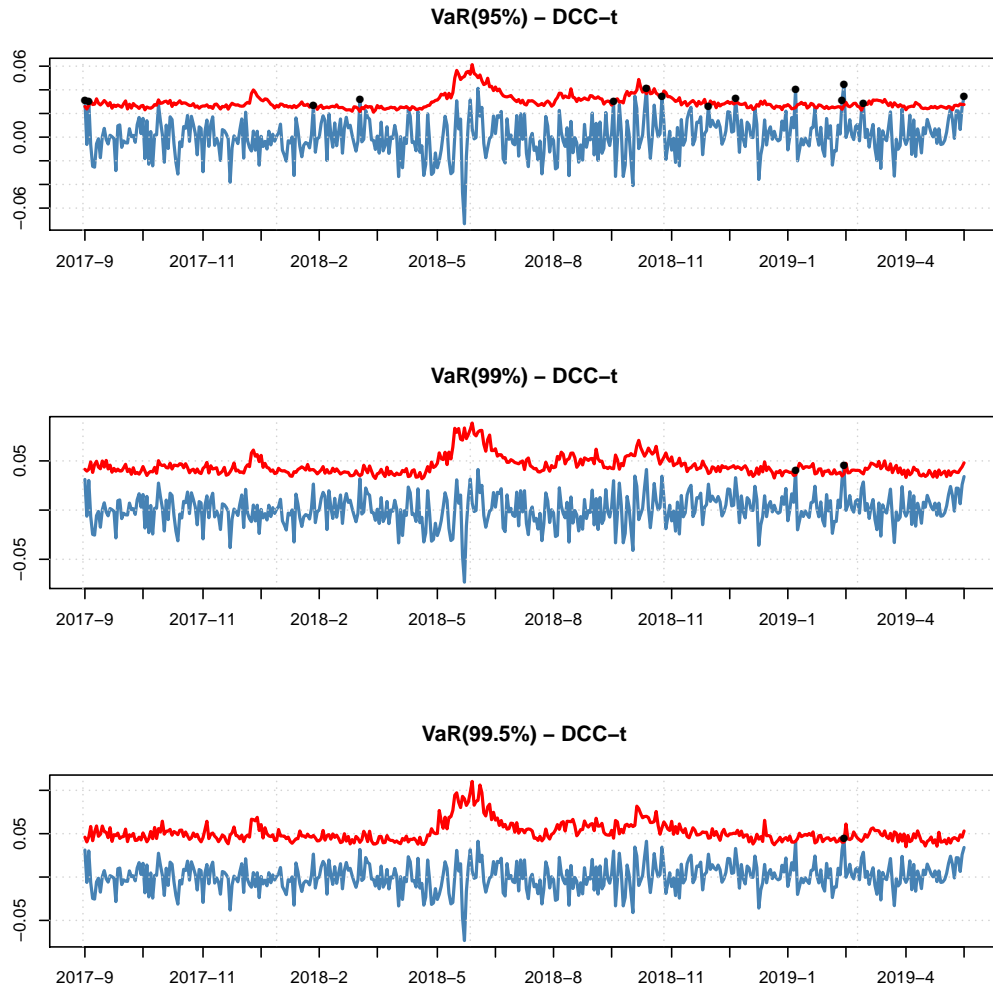


Figure F.3: Backtesting graphs for one-day ahead VaR of an equally weighted sovereign bond portfolio of ten countries and three maturities using a DCC model with t distribution. The upper panel shows the VaR with a 0.05 significance level, 0.01 for the middle panel and 0.005 for the bottom one. The solid line is the portfolio returns, the dotted line corresponds to the VaR and the solid circles are the exceptions.

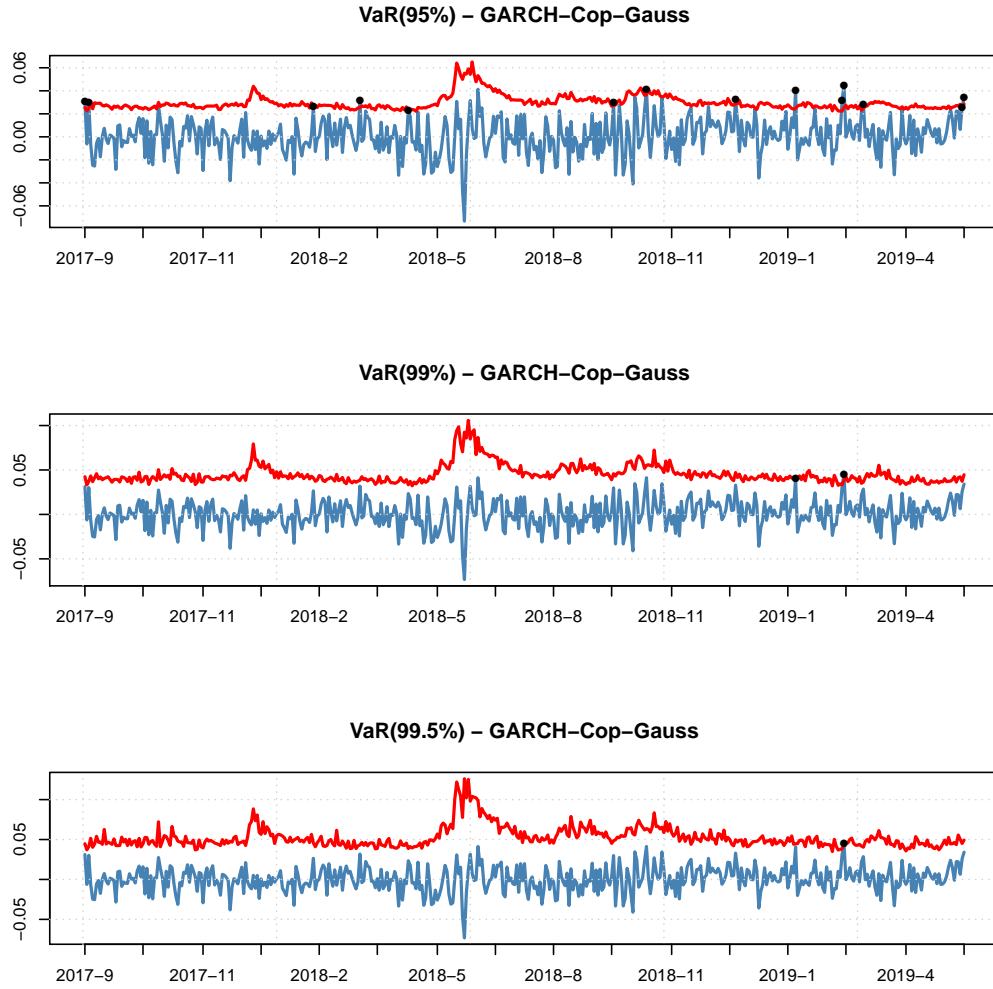


Figure F.4: Backtesting graphs for one-day ahead VaR of an equally weighted sovereign bond portfolio of ten countries and three maturities using a GARCH-Copula(Gaussian) model. The upper panel shows the VaR with a 0.05 significance level, 0.01 for the middle panel and 0.005 for the bottom one. The solid line is the portfolio returns, the dotted line corresponds to the VaR and the solid circles are the exceptions.

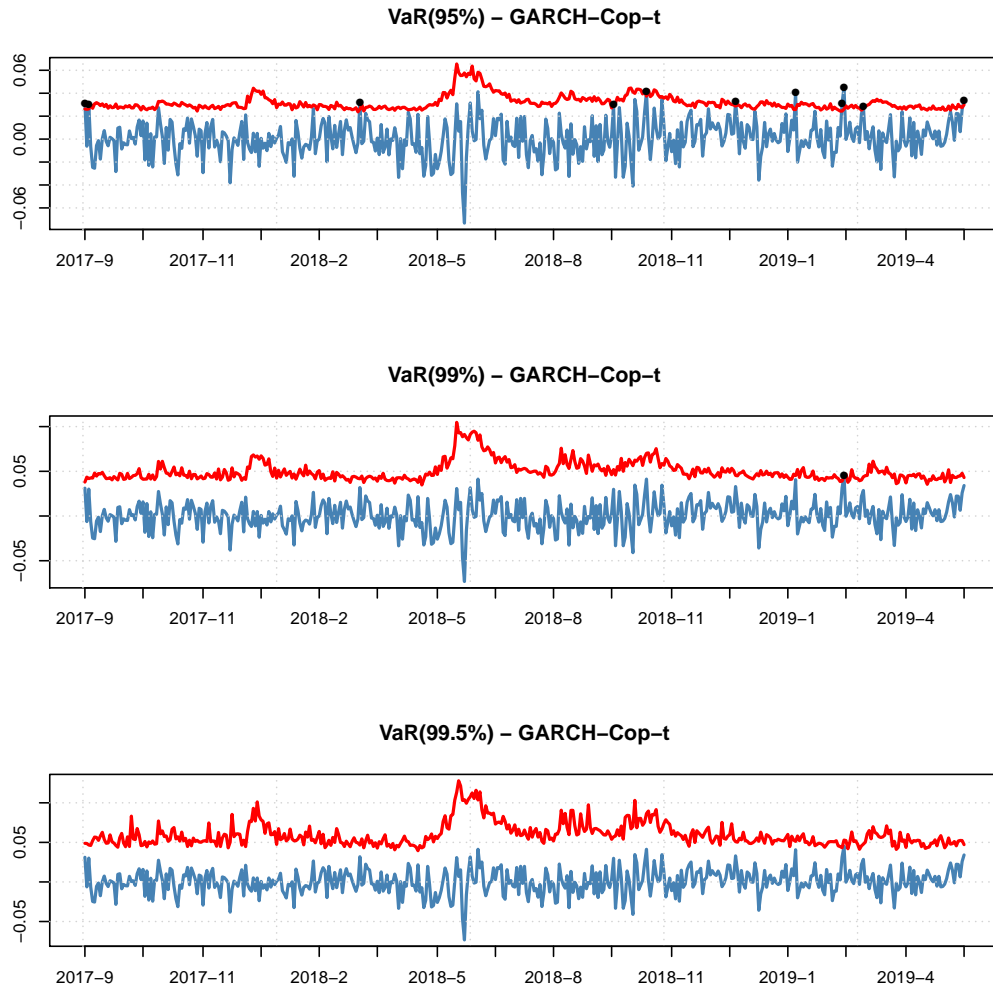


Figure F.5: Backtesting graphs for one-day ahead VaR of an equally weighted sovereign bond portfolio of ten countries and three maturities using a GARCH-Copula(t) model. The upper panel shows the VaR with a 0.05 significance level, 0.01 for the middle panel and 0.005 for the bottom one. The solid line is the portfolio returns, the dotted line corresponds to the VaR and the solid circles are the exceptions.

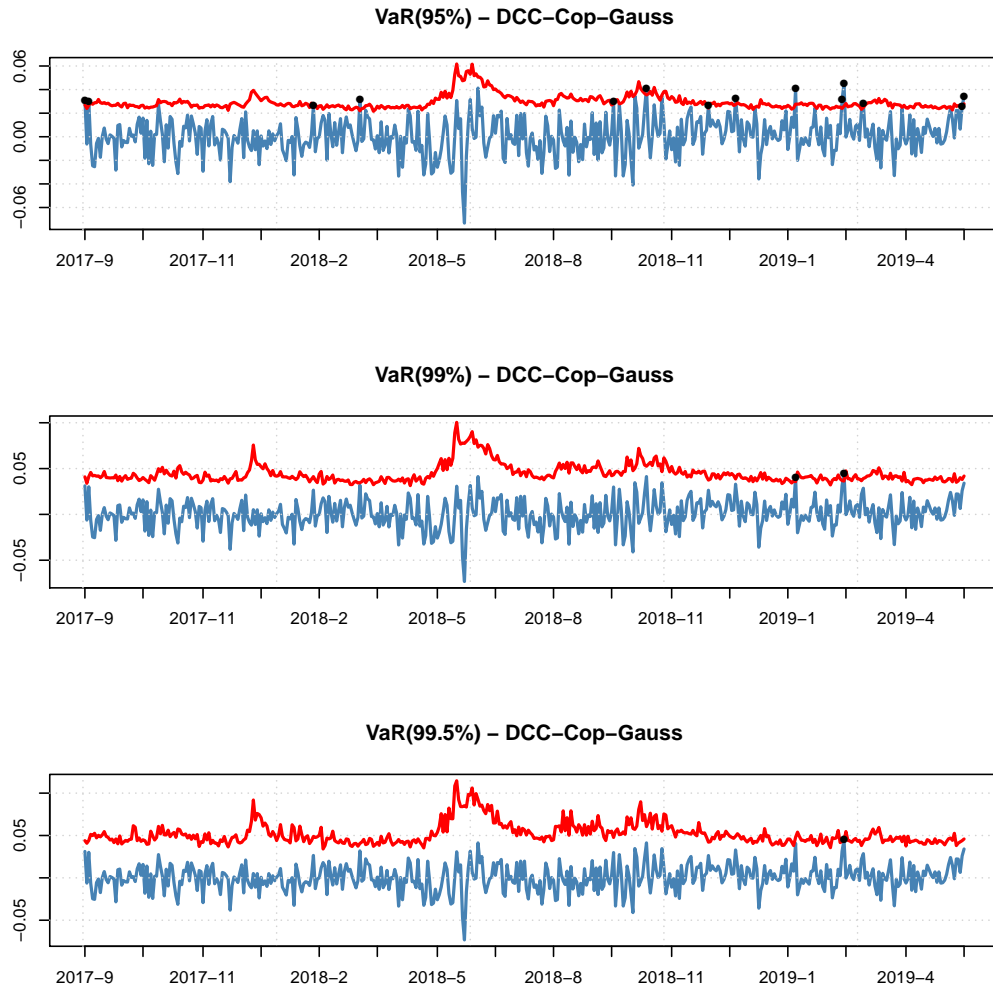


Figure F.6: Backtesting graphs for one-day ahead VaR of an equally weighted sovereign bond portfolio of ten countries and three maturities using a DCC-Copula(Gaussian) model. The upper panel shows the VaR with a 0.05 significance level, 0.01 for the middle panel and 0.005 for the bottom one. The solid line is the portfolio returns, the dotted line corresponds to the VaR and the solid circles are the exceptions.

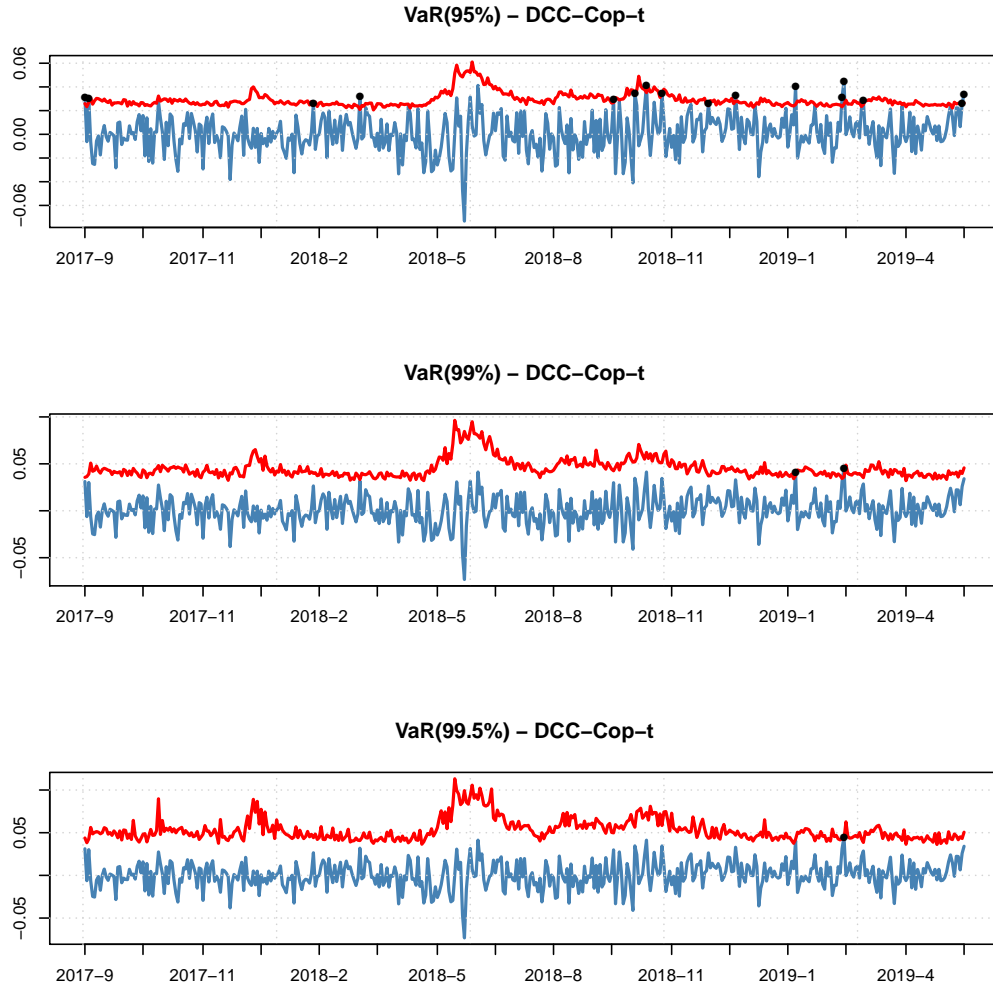


Figure F.7: Backtesting graphs for one-day ahead VaR of an equally weighted sovereign bond portfolio of ten countries and three maturities using a DCC-Copula(t) model. The upper panel shows the VaR with a 0.05 significance level, 0.01 for the middle panel and 0.005 for the bottom one. The solid line is the portfolio returns, the dotted line corresponds to the VaR and the solid circles are the exceptions.

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