A trade-off from the future: How risk aversion may explain the demand for illiquid assets*

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Abstract

We use a three-period model adopting a recursive definition of consumption to explore the optimal delegation that a present self, aware that her near-future self is present-biased but better informed, will make to protect her far-future self against income shocks. The model captures the present self's trade-off between using commitment mechanisms, restricting the near-future self's agency through illiquid savings, and profiting from the near-future self's better information about future shocks. Our main result states that agents with higher risk aversion can cover better against utility losses from time-inconsistent consumption through the commitment mechanism. Given the evidence of women being more risk-averse than men, this result provides the micro-foundation for the gender gap in adopting financial commitment devices, especially among single individuals.

Keywords: commitment devices; dynamic inconsistency; Epstein-Zin preferences; present bias

JEL codes: D11, D81, D90, G40

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1 Introduction

The standard consumption-savings problem focuses on the trade-off between present and future gratification. From a Behavioral Economics perspective, the interest in this problem dwells on the implications of dynamic inconsistencies, usually attributed to a lack of self-control; and the individual's awareness and strategies to overcome such self-control issues. Dynamic inconsistencies are modeled and measured through quasi-hyperbolic preferences (Angeletos et al. 2001, Frederick et al. 2002, Andreoni and Sprenger 2012*a*, Augenblick et al. 2015, Laibson 2015). Self-control, on the other hand, is incorporated through dynamic informational structures and variation in the agent's awareness of her present bias, defined as the tendency to prefer a smaller immediate reward instead of a larger later reward and a preference reversal when both rewards are delayed (Laibson 1997, O'Donoghue and Rabin 2001, DellaVigna and Malmendier 2004, 2006, Ali 2011, Galperti 2015, Chakraborty 2021).

Agents aware of their present bias tackle their lack of self-control through costly commitment mechanisms (Fudenberg and Levine 2006). These are chosen by a present self, at a cost to the near-future self (i.e., reducing consumption through savings), for the benefit of a far-future self (i.e., consuming what is saved). This three-period description of how commitment mechanisms work contrasts with the standard view of inter-temporal problems between present and future.

This paper unveils another trade-off, hidden in the future, between commitment and information. The present self has incentives to limit the agency of the near-future self to protect the far-future self from the near-future self's lack of control. This is the standard *commitment* channel. On the other hand, the present self would like to give more agency to the near-future self because she will have better information about prospective shocks that the far-future self may suffer. Since the near-future self cares about the far-future self, the present self knows that the former will pass more savings once a bad enough shock is anticipated. This is our proposed *informational* channel, which gives a leading role to risk preferences in understanding dynamic inconsistencies and ultimately offers alternative explanations to well-known empirical results about commitment devices. Since the present self aims to protect the far future self from a shock, the disutilities from altering the consumption stream are captured through risk aversion.

However, decisions resolved in the future are subject to inherent uncertainty and standard preferences representation confounding risk and time preferences (Halevy 2008, Andreoni and Sprenger 2012*b*). To see why the separation between risk and time is important, consider a traditional model with hyperbolic discounting capturing present-bias (Laibson 1997). The eagerness to smooth consumption across time is perfectly correlated with the eagerness to smooth consumption across states of nature. This confounding problem would limit the study of agents aware of their lack of self-control who are concerned about ensuring their future selves against shocks.

We disentangle risk from time preferences by adopting Epstein-Zin preferences (Epstein and Zin 1989, 1991) in a framework with dynamic inconsistency. These preferences, to which we will refer as EZ hereafter, are defined recursively over the known consumption in the present and a certainty equivalent of future utility. The recursive definition of EZ preferences breaks the indifference between a lottery yielding a high or a low consumption period after period, with a high- or low-value unknown until its realization; and a lottery defining an equivalent of a high or low consumption stream (as if one plays this lottery every period) that is resolved in the first period (Epstein and Zin 1989). With EZ preferences, the early resolution of uncertainty is preferred by agents whose risk aversion exceeds the inverse of the elasticity of intertemporal substitution.

To model the future trade-off between commitment and information, we add hyperbolic discounting to a three-period recursive problem employing EZ preferences. In the line of Fudenberg and Levine (2006), we adjust the narrative of the agent's awareness of her lack of self-control as if different selves have the leading role in each period. We start with the *present self*, who chooses an amount of investment in illiquid assets that she can only exchange for consumption in the far-future. This present self has full knowledge of events that will occur in the near-future, but her knowledge of the far-future is limited to a distribution of events. By contrast, the *near-future self* will have more knowledge of events that occur in the far-future and, with this information, she decides how much to save. The *far-future self* has a passive role and consumes what is left in this terminal period.

The difference between what the present- and the near-future self choose as consumption for the far-future self, contingent on the shock, triggers the dynamic inconsistency of interest. Thus, the present self decides an optimal delegation to the near-future self, balancing the losses from time-inconsistent consumption (i.e., the commitment channel) against her additional knowledge of the future (i.e., the information channel).

In principle, any channel can dominate the agent's action upon an increase in risk aversion. Our main result reveals that, in this case, the commitment channel dominates the informational one. Consequently, an increase in risk aversion results in a higher optimal investment in illiquid assets. For some intuition, remember that the near-future self passes fewer savings to the far-future than what the present self desires to be passed. Hence, contingent on negative shocks, the main problem faced by the present self is the discrepancy of the far-future consumption caused by her time inconsistency. Since more risk-averse individuals put more weight on unfavorable states of nature, the present self compensates by adding far future consumption in such states through illiquid savings. The present self "overrides" the near-future self's informational advantage as the utility loss from not covering these unfavorable states of nature is much more pronounced than the utility loss from restricting consumption in the near future.

Our result has implications when interpreting the empirical evidence documenting the demand for illiquidity in developing countries. Ashraf et al. (2006) reports an experiment in which almost two-thousand clients from a Philippine bank were offered a commitment savings product with the same interest rate as an alternative liquid savings account. The finding that women are more likely to take-up the commitment savings product is explained by the differences in roles within the household (i.e., women are more responsible for budgeting) and the awareness of being present-biased. Anderson and Baland (2002) report how individuals commit to investments in rotating savings and credit associations (ROSCAs) in Kenya. Although ROSCAs tackle the low access to formal financial services, their attractiveness and success partly depend on the willingness to reduce liquidity until the periodical investment is cashed-out, a commitment mechanism. They explain the disproportionately large female participation in ROSCAs as a consequence of intra-household differences in preferences for consumption and savings.

This finding offers a micro-foundation for the gender differences in the investment in illiquid assets. Several economic experiments have reported a greater risk aversion for women than for men (Byrnes et al. 1999, Eckel and Grossman 2008, Croson and Gneezy 2009, Charness and Gneezy 2012), reinforcing the dominance of the commitment channel over the informational channel. Ashraf et al. (2006) and Anderson and Baland (2002) report that single women are about 22-26 percentage points more likely to take-up the commitment mechanism than single men.[1] This gender gap among the single is harder to explain by intra-household differences and bargaining outcomes, reinforcing the role that differences in risk aversion may have.

This paper adds to the literature combining EZ preferences with hyperbolic discounting to understand the role of risk aversion in the use of commitment devices. It includes the study of social security (Fehr et al. 2008), savings decisions, and stock market participation

[1] See Ashraf et al. (2006), Table V, column 2.; and Anderson and Baland (2002), Table III, columns 1 and 2.

over the life cycle (Love and Phelan 2015). Moreover, our paper also speaks to the recent advances in experimental economics in separating the estimation of risk aversion, discount rate, and present bias (Andreoni and Sprenger 2012*a*, Andreoni et al. 2015).

The rest of this paper is organized as follows. Section 2 describes in detail the model. Section 3 finds the conditions that should be respected in equilibrium, yielding the determination of the optimal investment in illiquid assets. The comparative statics in risk aversion and the discussion of the monotonicity results are reported in Section 4. In Section 5, we drop some assumptions of our baseline model and study its implications. Section 6 concludes the article.

2 Model Set-up

We analyze the role of risk aversion in the demand for commitment devices for temporally inconsistent individuals. The model has three periods, indexed by $0, 1, 2$, and one *numeraire* homogeneous good. There are two types of assets, liquid and illiquid. To focus on preferences rather than the environment, we assume that the interest rate of both types of assets is the same. We assume that the utility in each period is strictly increasing in consumption, so uninvested assets are immediately consumed. The liquid asset can be consumed one period after its purchase, whereas consumption of the illiquid asset can only occur two periods after the investment. Hence, investment in illiquid assets can be regarded as a commitment device. To simplify the framework, we assume there is no possibility of borrowing, but it is straightforward to include this component.

We call self-t the individual's self at period $t \in \{0, 1, 2\}$. In each period, self-t chooses her consumption and the investment in liquid and illiquid assets. The three-period structure of the model implies that self-2 consumes any assets left, self-1 chooses between consumption and investment in liquid assets (i.e., she has no interest in buying illiquid assets because there is one period left), and self-0 distributes a total amount M between consumption, investment in liquid assets, and investment in illiquid assets. We denote $m \in [0, M]$ the total investment, and $s \in [0, m]$ the investment in illiquid assets. In Sections 3 and 4, we assume that m is exogenous (i.e., consumption of self-0 is given). In Section 5, we endogenize self-0's consumption.

Agents gain information about the future when they are one period behind the relevant time. Specifically, self-t has full information about the events in her near future, $(t + 1)$; but she knows only the distribution of the events in her far future, $(t + 2)$. Therefore, self-

0 knows that self-2 suffers a negative consumption shock K with a continuous probability distribution p, inducing a cumulative distribution P. Moreover, self-0 also knows that the shock suffered by self-2 will be deterministic for self-1, who has better information by being located one period ahead.

We consider a class of recursive preferences adapted from the EZ preferences (Epstein and Zin 1989). They allow us to disentangle risk aversion from intertemporal substitution. Let $f, g : \mathbb{R} \to \mathbb{R}$ be strictly increasing and strictly concave functions. Self-t has preferences about the consumption of self-τ, her future self in $\tau \geq t$. Given a random stream of consumption $C = (C_t)_t$, these preferences are represented by

$$V_\tau^t(C_\tau, C_{\tau+1}, \ldots) = R_\tau(Q_\tau^t(C_\tau, V_{\tau+1}^t(C_{\tau+1}, C_{\tau+2}, \ldots))),$$

where $R_\tau(X) = g^{-1}(\mathbb{E}_\tau[g(X)])$ for any random variable X, and

$$Q_\tau^t(x, y) = f^{-1}(f(x) + (\beta \mathbb{1}_{\{t=\tau\}} + \mathbb{1}_{\{\tau>t\}})f(y))$$

for any x, y in $\mathbb{R}$. As discussed in Epstein and Zin (1989), R_τ sets the risk preferences while Q_τ^t generates intertemporal preferences and time inconsistency. Note the g defined within R_t. We call it the risk-aversion-making function, which we will call the RAM hereafter. Intuitively, by increasing the concavity of g, we can account for the disutility of the uncertain state of the world in a given period. Applied to our three-period model, $V_0^0(C)$ represents the same preference relation on consumption streams as

$$\mathbb{E}_0[g \circ f^{-1}(f(C_0) + \beta f(C_1) + \beta f(C_2))].$$

For a better idea of how g operates, let us describe two scenarios in which this function plays no role, and our utility function collapses to the von Neumann-Morgenstern utility adapted to dynamic inconsistency (as introduced by Phelps and Pollak (1968)). First, when there is no within-period risk, the only source of consumption variation occurs over time. Hence, the function f aggregates the intertemporal certainty equivalent of consumption. Second, when the functions f and g are identical, the term $g \circ f^{-1}$ is the identity. Hence, the aggregation across time and states of nature becomes additive, respecting the expected utility hypothesis.

The EZ preferences generalize the von Neumann-Morgenstern utility by relaxing the axiom of independence over lotteries. This relaxation breaks the indifference between an uncertain consumption stream resolved at the beginning of the game or a consumption stream resolved period by period.[2] Therefore, g also shapes preferences for early resolution: the more concave g is, the stronger the preference for early resolution.

Finally, we discuss the role of the parameter β embedded in a δ-β-$\hat{\beta}$ model (O'Donoghue and Rabin 2001). In this model, δ captures the standard geometric discount rate, β imposes an additional discount to the stream of future utility, and $\hat{\beta}$ captures the agent's awareness of her lack of self-control. That is, the agent's estimation of her future self β. Time consistency implies that $\beta = \hat{\beta} = 1$, whereas a $\beta < 1$ captures the present bias by revealing a misalignment in the preferences of the present and the future selves.

A *sophisticated* agent is characterized by $\beta = \hat{\beta} < 1$, and a completely naive agent would be such that $\hat{\beta} = 1 > \beta$. To simplify, we set $\hat{\beta} = \beta$ and discuss the model for a sophisticated agent. Nonetheless, our results hold for $\beta < \hat{\beta} < 1$. We abstain from exploring the naive agent, as she will not demand the commitment mechanism.

We define β and $\hat{\beta}$ more precisely by setting, for self-t, the intertemporal marginal rate of substitution between periods τ and $\tau + 1$ as $\text{TMS}^t_{\tau,\tau+1}$. Self-0's investment decision depends on her own discount rate, $\text{TMS}^0_{0,1}$, but also on her belief about $\text{TMS}^1_{1,2}/\text{TMS}^0_{1,2} = \beta$, defined in $t = 0$ as $\hat{\beta}$.

3 Equilibrium

In our game, self-t takes decisions that self-$(t + 1)$ takes as given. Hence, we calculate the subgame perfect Nash equilibrium by backward induction. Self-2's consumption is the sum of all the liquid assets transferred by self-1 and the illiquid assets transferred by self-0. Hence, the first strategic interaction to consider is self-1's choice between consumption and liquid savings. When making this decision, self-1 has information about her near future, the shock with a cost k that self-2 will suffer; and about the past, the purchase s of illiquid assets made by self-0. For the sake of exposition, we set the interest rate to 0, but the results hold for non-zero interest rates. Thus, for a given investment in illiquid assets s made by self-0,

[2] As an example, take the *slap bet* from the "How I met your mother" sitcom: Lily gives Barney the choice of either ten immediate slaps or five slaps at any time. Barney chooses the five slaps, which Ted refers to as a "horrible call" because Barney must now live in constant fear of being slapped. Assuming they have EZ preferences, Ted seems to have a more concave utility function g, feeling relieved with the early resolution of uncertainty compared to Barney.

self-1's problem is given by

$$\max_{(x_1,x_2)\in\mathbb{R}^2} \{f(x_1)+\beta f(x_2)\}$$
$$\text{s.t.} \quad x_1 \le m-s,$$
$$x_2 \le m-c_1-k.$$

The two restrictions above imply that x_1 and x_2 can be negative. Hence, x_1 and x_2 can be interpreted as the deviation from a consumption path. With f being strictly increasing, the problem solved by self-1 is restricted to $x_2 = m-x_1-k$. The optimal consumption in period 1 for self-1, $c_1^1(s,k,m)$, is given by

$$c_1^1(s,k,m) = \begin{cases} m-s & \text{if } k < k_-(s,m), \\ c_1(k,m) & \text{if } k_-(s,m) \le k \end{cases}$$

where, for any k in the support of p, $m \in [0,M]$, and $s \in [0,m]$, $c_1(k,m)$ is the only solution of $f'(c_1(k,m)) := \beta f'(m-k-c_1(k,m))$ and $k_-(s,m) = s - f'^{-1}(\frac{1}{\beta} f'(m-s))$. The function k_- represents the minimum magnitude of a shock for which self-1 consumes less than what she has available (to increase self-2's consumption) for a given amount of total savings and illiquid savings.

Figure 1 gives some intuition on the implications of self-0's investment in illiquid assets. In each plot, the horizontal axis represents the economic magnitude of a negative shock for self-2, k (and negative values of k are gains). The left and right panels have three sections. The upper section depicts self-0's prior of the shock k, the middle section illustrates the dynamic inconsistency in consumption, and the bottom section displays self-0's utility loss caused by the difference between her actual (set by self-1) and optimal future consumption. We will use the notation of panel (a) to describe in detail the sections of this figure, though we will report the comparative analysis between panels later.

In the middle section, the time inconsistency is represented by the distance between the blue and the dashed black line. The blue line, denoted as $c_1(0,\cdot,m)$, represents self-1's desired consumption for herself in the absence of a liquidity constraint. The black line, denoted as $\tilde{c}_1$, represents self-0's desired consumption for self-1. The red dotted line, denoted as $c_1(s_a,\cdot,m)$, represents self-1's consumption constrained to self-0's illiquid savings s_a as a commitment mechanism.

Self-0's utility loss, depicted in the bottom section, gives the intuition of her optimization problem. Note that the minimum utility loss is reached when the black and red lines intersect each other and the magnitude of the shock is k_a. That is, when there is no difference between self-1's consumption and self-0's desired consumption for self-1. Given self-0's information availability, landing at this point is more a matter of luck. Nevertheless, we can think that the neighborhood around k_a is the region yielding the best protection against shocks suffered by self-2 from the viewpoint of self-0. Moving to the left in this neighborhood (i.e., when $k < k_a$), self-0 is harmed because she limited self-1's consumption too much while self-2 will have a positive income shock. Moving to the right in this neighborhood, when $k_a < k < k_-(s_a, m)$, self-0's utility loss comes from the mismatch between what she wanted that self-1 consumes and self-1's additional consumption. Nonetheless, the illiquid savings reduce self-1's consumption and attenuate self-0's utility loss. In the middle section, the second intersection of interest dwells between the blue and the red lines, when the shock is $k_-(s_a, m)$. Beyond this point, self-0's utility loss comes entirely from the dynamic inconsistency in consumption.

We finally compare panels (a) and (b), differing in self-0's investment in illiquid assets s_a and s_b, respectively, with $s_b > s_a$. Increasing the illiquid savings shifts downwards the intersection of the red line with the black and blue lines, displacing the neighborhood of shocks where self-0 is best at protecting self-2 toward expected larger shocks. Hence, an increase in illiquid savings increases the exposure to the utility loss from over-restraining self-1's consumption (under "good" shocks for self-2), while it reduces the exposure to the utility loss from shocks that would diminish self-2's available income.

As we will see in the next section, the optimal illiquid savings results from a trade-off between limiting self-1's agency to protect self-2's consumption given and expected shock, what we call the *commitment channel*; and maintaining self-1's agency since she has better information from the incoming shock, decreasing the chances from passing to the far future too much or too little. We call this the *informational channel*.

We close this section by formalizing self-0's optimization problem, who is aware that self-1's consumption depends on s. Hence, self-0 chooses s in order to maximize the objective function $\mathbb{E}_0[g \circ f^{-1}(f(c_0) + \beta f(c_1^1(s, K, m)) + \beta f(m - K - c_1^1(s, K, m)))]$, given by

$$\int_{k_-(s,m)}^{\infty} g(f^{-1}(f(c_0) + \beta f(c_1(k, m)) + \beta f(m - k - c_1(k, m))))\mathrm{d}P(k) \\ + \int_{-\infty}^{k_-(s,m)} g(f^{-1}(f(c_0) + \beta f(m - s) + \beta f(s - k)))\mathrm{d}P(k).$$

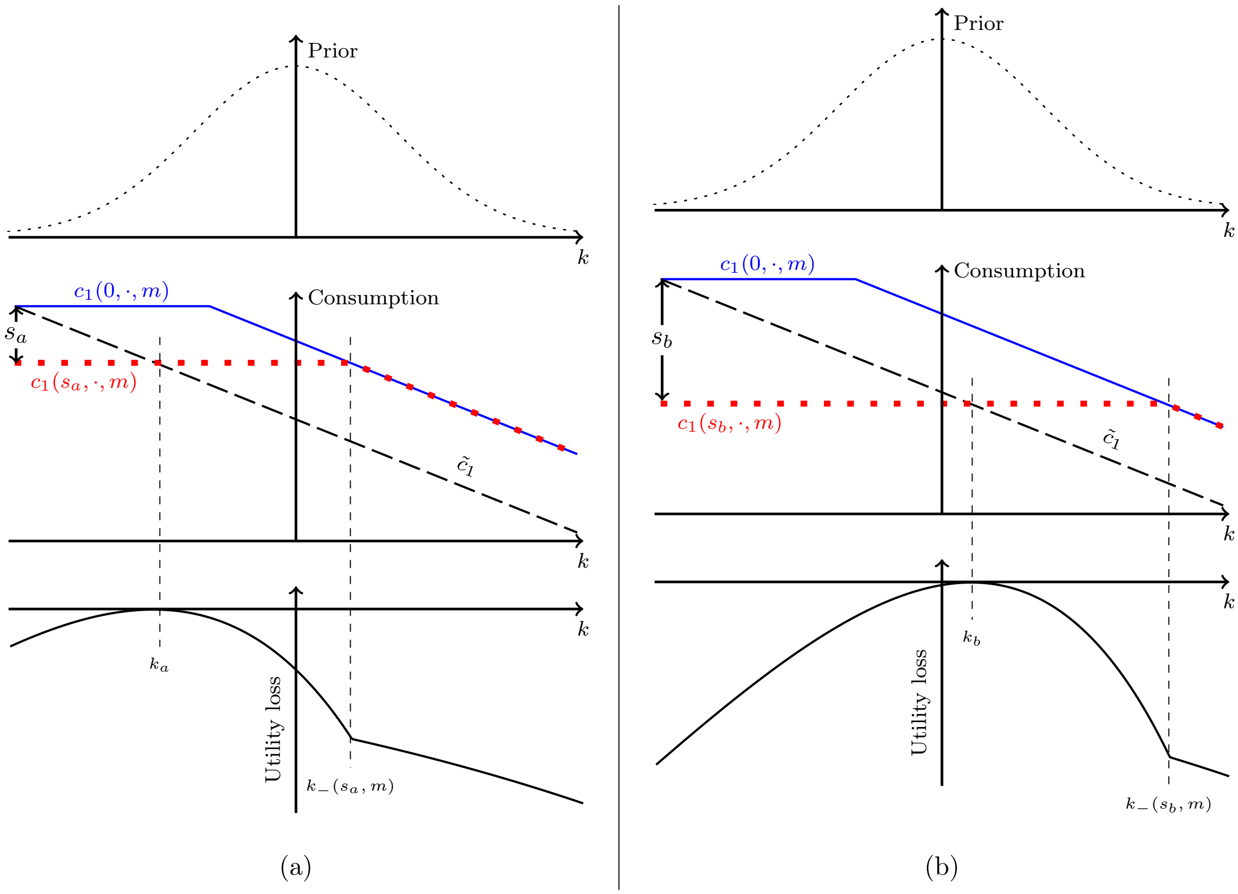

Figure 1: Illustration of self-0's decision problem based on the magnitude of the negative shock k (top section), time inconsistency in consumption and the corresponding commitment mechanism (middle section), and her trade-off (measured in utility loss) between restricting self-1's consumption decision too much and allowing the dynamic inconsistency to operate (bottom section).

Since P is continuous, the objective function is differentiable in s. Hence, for a fixed m, if the optimal investment $s^\star$ is interior (our case of interest), the first order condition given by

$$\int_{-\infty}^{k_-(s^\star,m)} \frac{g'(f^{-1}(f(c_0)+\beta f(m-s^\star)+\beta f(s^\star-k)))}{f'(f^{-1}(f(c_0)+\beta f(m-s^\star)+\beta f(s^\star-k)))}(-f'(m-s^\star)+f'(s^\star-k))\mathrm{d}P(k)=0 \quad (1)$$

must be respected.

When $\beta=1$ (i.e., a time-consistent or completely naive agent), the commitment channel disappears: self-0's and self-1's incentives are completely aligned. This is compatible with the non-existence of an interior solution in our model when there is no present bias. To see it analytically, suppose for a moment that there is an $s^\star$ solving Equation (1) with $\beta=1$. In this case, $k_-(s^\star,m)=-m+2s^\star$ and, for any k in the integration domain of Equation 1,

$s^\star - k > s^\star - k_-(s^\star, m) = m - s^\star$. Since f' is decreasing, $f'(m - s^\star) > f'(s^\star - k)$ and $f'(m - s^\star) - f'(s^\star - k) < 0$. But $f', g' > 0$, so the integrand in Equation (1) is non-positive. Since P is continuous, the integral is negative (see Figure 2). This contradicts the statement that $s^\star$ solves the first order conditions. Moreover, since the integrand in Equation (1) is always negative when $\beta = 1$, the optimal investment is the minimum possible one, and self-0 does not invest. A time-consistent self would not limit the choices that her future self may make with better information.

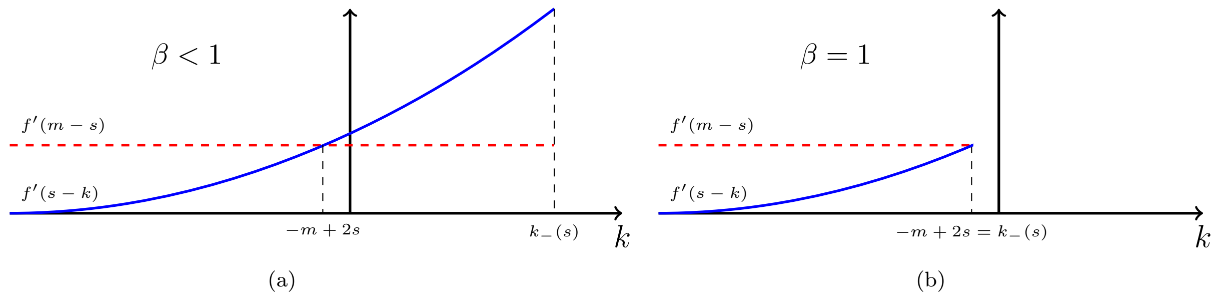

Figure 2: Panels compare $f'(m - s)$ and $f'(s - k)$. In panel (a), $\beta < 1$, and thus, it might be possible to respect the first order conditions of self 0. In panel (b), $\beta = 1$, and it is impossible to meet the first order conditions.

4 Comparative Static on Risk Aversion

In this section, we examine the effect of risk aversion on the take-up of illiquid assets. Imagine two agents, 1 and 2, with RAM g_1 and g_2, respectively. We say that agent 1 is more risk averse than agent 2 if $-\frac{g_1''}{g_1'} > -\frac{g_2''}{g_2'}$. Our comparative static is non-parametrized: we start from preferences defined by functional forms f and g, as in Section 2, and we study changes in the equilibrium as the risk aversion increases. We keep fixed the function f defining intertemporal preferences.

Let $\mathcal{C}$ be the set of increasing and strictly concave functions mapping $\mathbb{R}$ onto $\mathbb{R}$. Note that, if $h \in \mathcal{C}$, then an individual with a RAM $h \circ g$ is more risk averse than an individual with a RAM g.[3] We set $U(s, m; g)$ as the utility of an agent with RAM g that purchased s on illiquid and $m - s$ on liquid assets (recall that m is self-0's total investment in future consumption). We also define $\{s_g^\star(m)\} := \text{argmax}_{s \in [0,m]}\{U(s, m; g)\}$ whenever the solution is unique.

[3]We have $-(h \circ g)''/(h \circ g)' = -(h''(g)g'^2 + h'(g)g'')/(h'(g)g') = -g'[h''(g)/h'(g)] - g''/g'$. Since h is in $\mathcal{C}$, the term $h''(g)/h'(g)$ is negative and we obtain $-(h \circ g)''/(h \circ g)' > -g''/g'$.

We present the first main result of this paper.

Proposition 1. *Fix m and let $h \in \mathcal{C}$. Suppose that, for each $\lambda \in [0,1]$, there is a unique interior solution $s^\star_{\lambda(h\circ g)+(1-\lambda)g}(m)$ for the problem of maximizing $U(\cdot, m; \lambda(h \circ g) + (1-\lambda)g)$. Then, $s^\star_{h\circ g}(m) > s^\star_g(m)$.*

In simpler words, if agent 1 is more risk averse than agent 2, and if any agent with a risk aversion between theirs has a unique (interior) optimal take-up of illiquid assets, then agent 1 invests more in illiquid assets than agent 2.

Section 3 introduced the trade-off between the commitment and informational channels in self-0's decision. Whereas the former induces self-0 to save more on illiquid assets, the latter pushes self-0 to delegate more decision power to the better-informed self-1, inducing more savings on liquid assets. Intuitively, a more risk-averse self-0 is more afraid of self-1's additional and time-inconsistent consumption. However, self-0 also values the additional information that self-1 will obtain about k. Proposition 1 states that, as the risk aversion increases, the magnitude of the commitment channel increases more than the informational one.

Figure 3 gives an intuition for Proposition 1. Panel (a) combines the top and bottom sections of Figure 1 for self-0's optimal illiquid investment, $s^\star_g$, given self-0's prior with respect to k (i.e., the dotted distribution on top). Recall that the solid curve displays self-0's utility loss, contingent on k, due to the differences in the actual self-1's consumption and self-0's desired one. This curve has a kink at $k_-(s^\star_g, m)$. Moreover, we have colored the regions divided by this kink, which separates the operation of the commitment and the informational channels. Self-0's commitment channel operates in the purple area by limiting self-1's consumption deviation from self-0's desires. Nevertheless, self-0 is aware that, if the expected shock is sufficiently large, self-1 will use the additional information regarding the expected losses to increase self-2's protection. Hence, the informational channel partially aligns self-0 and self-1, causing this kink in which the marginal utility loss is reduced. Self-0's trade-off can be considered the optimal delegation to self-1 to protect regions subject to relatively high probabilities of suffering high losses, while reducing the harm of dynamic inconsistencies in consumption if the shock is not very large.

Panel (b) lets us observe the consequences of setting $s^\star_g$ as the amount of illiquid savings for individuals with RAM more concave than g (i.e., with a greater risk aversion). The dotted and solid curves are the same from Panel (a), after shrinking the vertical axis for illustration purposes. Each curve below the solid one represents a utility loss for agents with increasing distinct levels of risk aversion for the same $s^\star_g$. The higher the risk aversion, the lower the

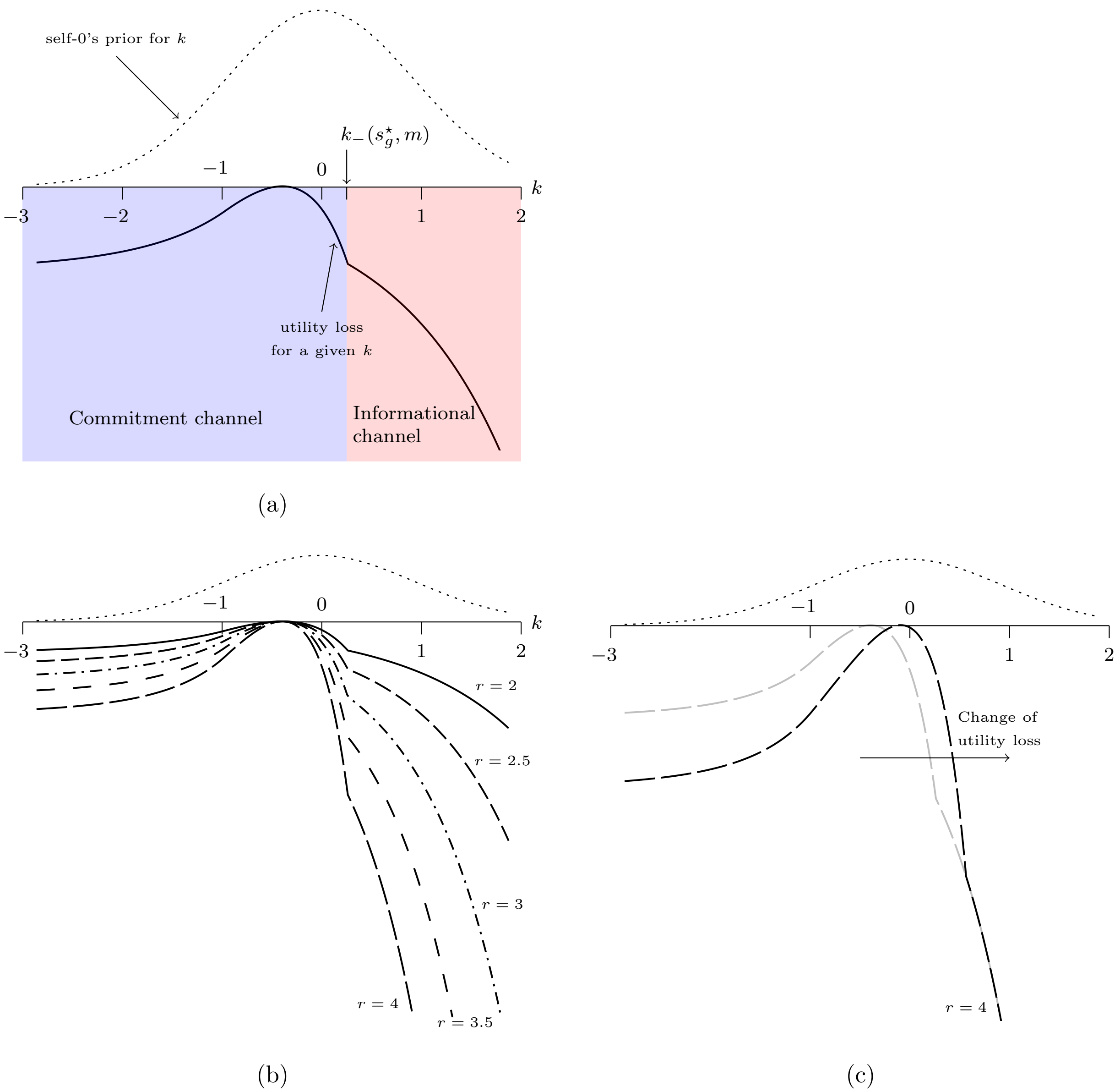

Figure 3: Loss of utility of self 0 as a function of the utility loss suffered by self 2. The exogenous consumption at period 0 is $c_0 = 0$, the total wealth is $m = 1$, $f = -\exp(-(\cdot))$, $g = -\exp(-2(\cdot))$, the shock is Normal with mean 0 and variance 1, and the investment $s = 0.302$ is optimal for these parameters. Panel (b) presents the utility loss when $s = 0.302$ for higher absolute risk aversions functions, all constant, ranging from 2 to 4. Panel (c) depicts how the utility loss changes for an agent with $r = 4$ when savings changes from 0.302 (the optimal for an agent with $r = 4$) to 0.462 (the optimal for the agent with $r = 4$).

dashed curve. Note that the kink does not move horizontally because self-1's consumption is limited to be at most $m - s_g^\star$ and decides consumption after knowing the shock. Moreover, if self-1's consumption is fixed and self-1 is not subject to uncertainty, the vertical drop in the kink's position captures the utility loss embedded in the risk preferences that are associated with the risk-free time inconsistency in consumption.

The asymmetry to the left and right of the bliss point, where self-0's utility loss is minimized, holds the intuition for why a more risk-averse self-0 will increase her investment in illiquid assets, as depicted in Panel (c). Note that, due to the concavity of the preferences, the marginal utility loss when moving away from this bliss point is larger in scenarios of scarcity (i.e., with a greater k) than in scenarios of abundance (i.e., with a negative k implying a positive shock). Hence, self-0 gains from shifting rightward this bliss point through more illiquid savings, as it displaces the scenarios of larger utility losses to regions with a lower probability of occurrence. In terms of the mechanisms, the commitment channel gains importance relative to the information channel because, as risk-aversion increases, self-0 becomes so concerned about unexpectedly large shocks that she directly protects self-2.

5 Extensions

In this section, we briefly discuss the consequences of separately dropping two of the simplifying hypotheses employed in Section 2: adding a positive interest rate i to the investments and making self-0's consumption endogenous.

Assets with Interest Rate

We explore the interplay between the commitment and informational channels with risk aversion under a non-null interest rate. We omit the straightforward algebra and focus on the main intuitions.

Let us fix an identical interest rate for the near and far future. The interest returns in the near-future will reinforce the information channel, whereas its returns in the far future will reinforce the commitment channel. The latter reinforcement is evident since interest returns in the far future protect self-2 from shocks. For the former, we dedicate a couple of extra lines. Interest returns in the near-future reduce the cost of displacing consumption to the far-future. Since self-1 has better information than self-0, she can implement better consumption levels for self-2 under "bad" shocks, reinforcing the informational channel. The question is which of the reinforced channels dominate, and we find that it depends on the size of the interest rate. When the interest rates are low, or even negative, scarcity makes self-0 to directly protect self-2 through more investment in illiquid assets (as discussed in the previous section). By analogy, the informational channel dominates if the interest rate is high.

The conclusions of Section 4 do not depend on the interest rate, so fixing the interest rate and allowing the risk aversion to vary, we get that more risk-averse individuals save more in illiquid assets. Figure 4 summarizes all the elements of our precedent analysis and presents the optimal take-up of illiquid assets for several interest rates and levels of risk aversion. This figure also displays a seemingly counterintuitive behavior: the demand for illiquid savings decreases when the interest rates increase. Nevertheless, this pattern is consistent with the dominance of the informational channel as the interest rate increases. Self-0 is more willing to delegate more informed decisions to self-1 when the interest rates from liquid savings reduce resource scarcity.

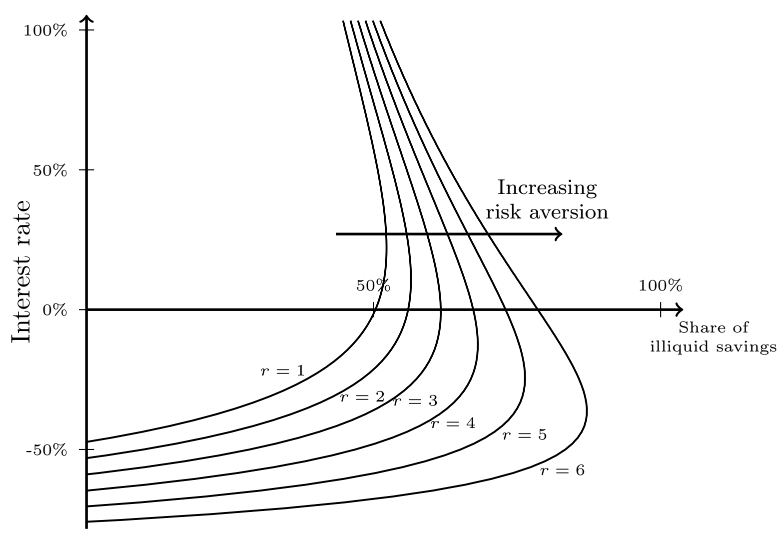

Figure 4: Optimal proportion of illiquid savings with respect to the total savings as a function of the interest rate for several levels of risk aversion. The cost distribution is normal, with an average of 0 and a variance of 1/4. We set $M = 1$, $\beta = 1/2$, $f = -\exp(-(\cdot))$ and $g = -\exp(-r(\cdot))$ for $r = 1, 2, \ldots, 6$.

Endogenous Consumption in Period 0

So far, we modeled self-0's saving decision in liquid and illiquid assets while taking her consumption as given. In this section, we relax this hypothesis and consider m endogenous.

To begin this analysis, let

$$\phi(m,c,k) := f^{-1}(f(M-m)+\beta f(c)+\beta f(m-c-k))$$

for each $(m,c,k) \in [0,M]\times\mathbb{R}^2$. Suppose the solution $(m_g^\star, s_g^\star)$ of an agent with RAM g that maximizes $U(s,m;g)$ for $m\in[0,M]$ and $s\in[0,m]$ is interior. Then, together with Equation 1 (replacing $s^\star$ by $s_g^\star$ and m by $m_g^\star$, of course) defining the first order condition related to s, we have a first-order condition related to m. It is given by

$$\int_{k_-(s_g^\star,m_g^\star)}^{\infty} \frac{g'(\phi(m_g^\star, c_1(k,m_g^\star),k))}{f'(\phi(m_g^\star, c_1(k,m_g^\star),k))}(-f'(M-m_g^\star)+\alpha(k,m_g^\star)f'(c_1(k,m_g^\star)))\mathrm{d}P(k) \\ + (-f'(M-m_g^\star)+\beta f'(m_g^\star - s_g^\star))\int_{-\infty}^{k_-(s_g^\star,m_g^\star)} \frac{g'(\phi(m_g^\star, m_g^\star - s_g^\star, k))}{f'(\phi(m_g^\star, m_g^\star - s_g^\star, k))}\mathrm{d}P(k) = 0, \quad (2)$$

where

$$\alpha(k,m) := 1-(1-\beta)\frac{\beta f''(m-k-c_1(k,m))}{\beta f''(m-k-c_1(k,m))+f''(c_1(k,m))},$$

which, after some algebra, can be shown to be in $[\beta,1]$ for each k,m. We also introduce the following assumption.

Assumption 1. *The function f is twice-differentiable, and its absolute risk aversion coefficient is decreasing and convex.*

We recall that f is related to the intertemporal preferences in our model, so the risk aversion coefficient is mathematically defined but, in this case, not related to uncertainty. This assumption still leaves us with a relatively large set of functions employed in Economics to model the utility, and, in particular, it includes the CRRA and CARA functions.

We also assume that the cross derivative of $U(\cdot,\cdot;\lambda(h\circ g)+(1-\lambda)g)$ is positive for each $\lambda\in[0,1]$. If it is not valid in the whole domain, it holds at least at $(s^\star_{\lambda(h\circ g)+(1-\lambda)g}, m^\star_{\lambda(h\circ g)+(1-\lambda)g})$. We have two remarks about this condition. First, thanks to the second order condition, the cross derivative at the maximum is lower-bounded (by a value that can be negative), say L_1. Second, it is possible to weaken the cross derivative assumption and allow it to be negative, respecting a lower bound, say L_2, depending on the agent's preferences. In the generic case, we could not prove that $L_2 > L_1$ (which would convert the assumption to a lemma).

Proposition 2. *Suppose that the problem of maximizing $U(\cdot;\lambda(h\circ g)+(1-\lambda)g)$, for each $\lambda\in[0,1]$, has a unique interior solution $(s^\star_{\lambda(h\circ g)+(1-\lambda)g}, m^\star_{\lambda(h\circ g)+(1-\lambda)g})$. Then, under Assumption 1 and the cross derivative assumption, we have $s^\star_{h\circ g} > s^\star_g$ and $m^\star_{h\circ g} > m^\star_g$.*

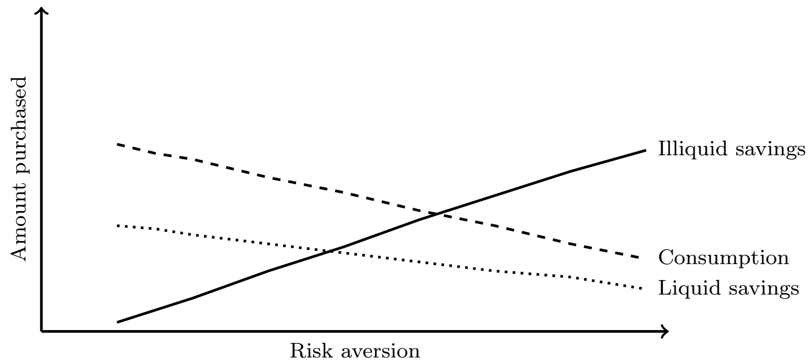

Figure 5: Optimal amounts of illiquid savings, illiquid savings, and consumption at period 0 for several levels of risk aversion. The cost distribution is normal, with an average of 1/3 and a variance of 1. We set $M = 1$, $\beta = 1/2$, $f = -\exp(-(\cdot))$ and $g = -\exp(-r(\cdot))$ for $r = 0.5, 1, 1.5, \ldots, 4$.

A simple way of stating this proposition is that, under the imposed assumptions, a more risk-averse individual will pass more assets to the future and, in addition, increase her investment in illiquid assets.

Making m endogenous implies that more risk-averse agents decrease their disutility from time-inconsistent consumption streams either by increasing m (i.e., by sacrificing present consumption), or at least by increasing s (i.e., by purchasing more illiquid assets). It reinforces the dominance of the commitment over the informational channel.This is true even without the cross derivative assumption.

By adding the cross derivative assumption, an increase in m will also increase the marginal benefits from s (and vice versa), making Proposition 2 to hold. Since it is a strong assumption, it may occur that the space of parameters yielding an increase in the take-up of illiquid assets after increasing risk aversion is empty. Figure 5, plotting the optimal choices as a function of risk aversion when consumption of self-0 is endogenous, reveals that this is not the case. We observe the same implications of Proposition 2, where an increase in risk aversion increases investment in illiquid savings.

6 Conclusion

In this paper, we present the trade-off between a commitment mechanism and information acquisition in a dynamic consumption problem subject to present bias. By considering the time-inconsistency problem to a three-period model representing the present-, near future-, and far future- selves; we aim to understand the present self's optimal delegation level, in terms of assets, to the present-biased though better informed near future-self. The far-future self will face a shock for which the present self has less information than the near-future self. Hence, if the present self invests too much in illiquid assets directly passed to the far-future she may restrict the near-future self unnecessarily, causing a utility loss from a stringent use of the commitment mechanism. By contrast, if the present self invests too little in illiquid assets, benefiting from the near-future self's additional information, she will have a utility loss from the disagreement in the future selves' consumption.

The main challenge in this model comes from the need to separate time from risk preferences. Since the future is inherently uncertain, this separation allows us to distinguish disutility losses related to time-inconsistent consumption streams from the disutility losses from a shock for which we gain information as we delve into the future. We tackled this challenge by using Epstein-Zin preferences (Epstein and Zin 1989, 1991), defined recursively over the known consumption in the present and a certainty equivalent of future utility.

Our main result reveals that, as risk-aversion increases, the trade-off between the commitment and the information channels is dominated by the former. The reason is that more risk-averse preferences imply a disproportionately larger utility loss in case of strong "bad" shocks, yielding larger discrepancies between self-0's desired and actual consumption. As a consequence, the present self invests more in illiquid savings. This result, combined with the incentivized measurements from Experimental Economics revealing higher risk aversion among females (Charness and Gneezy 2012), provides a micro-founded explanation for the higher take-up of formal and informal financial commitment devices in developing countries (Anderson and Baland 2002, Ashraf et al. 2006). Whereas the most prominent explanations for the gender gap in the take-up of such products were related to intra-household differences and bargaining outcomes, our results can explain why this gap is also prevalent among single women and men.

Economic paradigms jointly exploring risk and time preferences revealed strong preferences for certainty, regardless of whether an uncertain alternative is offered ahead or behind in the temporal line (Halevy 2008, Andreoni and Sprenger 2012*b*). Our results motivate further empirical explorations of allocations between future periods subject to income shocks (e.g., employing convex time budgets for future payments with a fixed show-up payment) while paying special attention to gender differences.

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Lemmata and Proofs of Propositions

Lemma 1. *Let $a, b \in \overline{\mathbb{R}}$ with $b > a$. Assume u, v, and t are continuous real functions on $[a, b]$, that $u > 0$, that v and t are increasing, and that $t \geq 1$. Define*

$$A := \int_a^b u(x)v(x)\mathrm{d}P(x) \geq 0.$$

Then

$$\int_a^b t(x)u(x)v(x)\mathrm{d}P(x) \geq A.$$

In particular, if there is $y \in [a, b]$ such that $t(y) > 1$, then the last the inequality is strict.

Proof. If $v(a) \geq 0$, then $t(x)u(x)v(x) \geq u(x)v(x)$ for each $x \in (a, b)$ and the inequality holds trivially. Assume that $v(a) < 0$. Since v is continuous and $A \geq 0$, for some $\widetilde{x} \in (a, b)$ we must have $v(\widetilde{x}) = 0$. Because $t(x) \leq t(\widetilde{x})$ and $v(x) \leq 0$ for each $x \leq \widetilde{x}$, the inequality $\int_a^{\widetilde{x}} t(\widetilde{x})u(x)v(x)\mathrm{d}P(x) \leq \int_a^{\widetilde{x}} t(x)u(x)v(x)\mathrm{d}P(x)$ must hold. Analogously, we have $\int_{\widetilde{x}}^b t(\widetilde{x})u(x)v(x)\mathrm{d}P(x) \leq \int_{\widetilde{x}}^b t(x)u(x)v(x)\mathrm{d}P(x)$. Observing that $t(\widetilde{x}) \geq 1$ gives

$$A \leq \int_a^b t(\widetilde{x})u(x)v(x)\mathrm{d}P(x) \leq \int_a^b t(x)u(x)v(x)\mathrm{d}P(x),$$

concluding the proof. □

Lemma 2. *If $h \in \mathcal{C}$, then*

$$\partial_1 U(s_g^\star(m), m; h \circ g) > 0.$$

Proof. Define, for each real k, $\psi(k) := h'(g(\phi(m, m - s_g^\star(m), k)))$. It is easy to verify that ψ is increasing. Define $\psi(-\infty) := \lim_{k \to -\infty} \psi(k) > 0$. Now, define $t(k) := \psi(k)/\psi(-\infty) > 1$, $v(k) := -f'(m - s_g^\star(m)) + f'(s_g^\star(m) - k)$, increasing in k, and $u(k) := \frac{g'(\phi(m, m - s_g^\star(m), k))}{f'(\phi(m, m - s_g^\star(m), k))} > 0$. Thus, we can apply the chain rule and Lemma 1 to obtain

$$\partial_1 U(s_g^\star(m), m; h \circ g)/\psi(-\infty) = \int_{-\infty}^{k_-(s_g^\star(m))} t(k)u(k)v(k)\mathrm{d}P(k) > \int_{-\infty}^{k_-(s_g^\star(m))} u(k)v(k)\mathrm{d}P(k) = 0.$$

We get the desired result by noticing that $\psi(-\infty) > 0$. □

For Lemma 3 and the proof of Proposition 1, whenever $s_g^\star(m)$ is well-defined, for each $g, h \in \mathcal{C}$, $m \in [0, M]$, and $\lambda \in [0, 1]$, we set

$$Y_s(\lambda; m, g, h) := \partial_1 U(s_g^\star(m), m; \lambda(h \circ g) + (1 - \lambda)g).$$

We also define Y'_s as the derivative with respect to the first argument of Y_s.

Lemma 3. *Let $h \in \mathcal{C}$ be strictly concave and assume that $s^\star_{\lambda(h\circ g)+(1-\lambda)g}(m)$ is unique and interior for each $\lambda \in [0,1]$. Then, $Y'_s(0; m, \lambda(h\circ g) + (1-\lambda)g, h) > 0$.*

Proof. Since h is strictly concave and g is concave, the function $\mu(h \circ g) + (1-\mu)g$ is strictly concave. Observe that

$$Y_s(\mu; m, g, h) - Y_s(0; m, g, h) = \mu\partial_1 U(s^\star_g(m), m; h\circ g) - \mu\partial_1 U(s^\star_g(m), m; g),$$

so

$$\begin{aligned} Y'_s(0; m, g, h) &= \lim_{\mu\to 0} \frac{Y_s(\mu; m, g, h) - Y_s(0; m, g, h)}{\mu} \\ &= \lim_{\mu\to 0} \frac{\mu(\partial_1 U(s^\star_g(m), m; h\circ g) + \partial_1 U(s^\star_g(m), m; g))}{\mu} \\ &= \partial_1 U(s^\star_g(m), m; h\circ g) + \partial_1 U(s^\star_g(m), m; g). \end{aligned}$$

But we know that $\partial_1 U(s^\star_g(m), m; g) = 0$ (this is the first order condition) and Lemma 2 ensures that $\partial_1 U(s^\star_g(m), m, (h\circ g)) > 0$, so $Y'_s(0, m; g, h) > 0$. Now, repeat the lines above replacing g by $\lambda(h \circ g) + (1-\lambda)g$, and, since it holds for any $\lambda \in [0,1]$, we immediately find the desired result. □

Proof of Proposition 1

Proof. Let $\widehat{g}_\lambda := \lambda(h\circ g) + (1-\lambda)g$ and $\widehat{s}(\lambda) := s^\star_{\widehat{g}_\lambda}(m)$. Since $\widehat{s}(\lambda)$ is a global interior maximum of $U(\cdot, m, \widehat{g}_\lambda)$ and $U(\cdot, \widehat{g}_\lambda)$ is differentiable at $(\widehat{s}(\lambda), m)$, we have $\partial_1^2 U(\widehat{s}(\lambda), m; \widehat{g}_\lambda) < 0$. Thus, we can apply the Implicit Function Theorem (IFT) to obtain

$$\widehat{s}'(\lambda) = -Y'_s(0; m, \widehat{g}_\lambda)/\partial_1^2 U(\widehat{s}(\lambda), m; \widehat{g}_\lambda).$$

Lemma 3 tells us that $Y'_s(0; m, \widehat{g}_\lambda) > 0$, so $\widehat{s}'(\lambda) > 0$ and, applying the Fundamental Theorem of Calculus, we obtain

$$s^\star_{h\circ g}(m) - s^\star_g(m) = \int_0^1 \widehat{s}'(\lambda)\mathrm{d}\lambda > 0,$$

which concludes the proof. □

Lemma 4. *We have $-f'(M - m^\star_g) + \beta f'(m^\star_g - s^\star_g) < 0$.*

Proof. First, observe that α is in $[\beta, 1]$ for each k, m since $\frac{\beta f''(m-k-c_1(k,m))}{\beta f''(m-k-c_1(k,m))+f''(c_1(k,m))} \in [0,1]$. Since $c_1(k,m) \leq m_g^\star - s_g^\star$, f' is decreasing, and $\alpha \geq \beta$, we have that

$$-f'(M - m_g^\star) + \alpha(k, m_g^\star) f'(c_1(k,m)) > -f'(M - m_g^\star) + \beta f'(m_g^\star - s_g^\star).$$

Assume that $-f'(M - m_g^\star) + \beta f'(m_g^\star - s_g^\star) \geq 0$. This would imply that the first integral of Equation 2 is strictly positive and that the second one is non-negative, a contradiction, since the sum of the integrals is 0. □

Lemma 5. *We have*

$$\partial_1 c_1(k,m) = -\partial_2 c_1(k,m) = \frac{-\beta f''(m-k-c_1(k,m))}{\beta f''(m-k-c_1(k,m)) + f''(c_1(k,m))} < 0.$$

As a consequence, $-1 < \partial_1 c_1 < 0$ *and* $0 < \partial_2 c_1 < 1$.

Proof. Define $\psi(c,k) := f'(c) - \beta f'(m-k-c)$ for each $c, k \in \mathbb{R}$, we have that $\partial_1 \psi(c,k) = f''(c) + \beta f''(m-k-c) \neq 0$, so we can apply the IFT to obtain the desired result. □

Lemma 6. *If Assumption 1 holds, then* $\alpha(\cdot, m)$ *is increasing for each* $m \in \mathbb{R}$.

Proof. We have $\alpha(k,m) = 1 + (1-\beta)\partial_1 c_1(k,m)$, so it is enough to show that $\partial_1 c_1(\cdot, m)$ is increasing. To shorten the notation, we set $\tilde{f} := f(m-k-c_1(k,m))$ (and the analogous for the derivatives). First, using Lemma 5 to show that $\partial_1^2 c_1(k,m)$ equals

$$\frac{\beta((1+\partial_1 c_1(k,m)) f''(c_1(k,m)) \tilde{f}''' + f'''(c_1(k,m)) \partial_1 c_1(k,m) \tilde{f}'')}{(\beta f''(m-k-c_1(k,m)) + f''(c_1(k,m)))^2}.$$

Since β and the denominator are positive, we only look at the term in the parenthesis in the numerator. We use Lemma 5 again to eliminate the term $\partial_1 c_1(k,m)$, and we have left

$$\frac{(f''(c_1(k,m)))^2 \tilde{f}''' - \beta f'''(c_1(k,m)) (\tilde{f}'')^2}{\beta f''(m-k-c_1(k,m)) + f''(c_1(k,m))}.$$

Using the identity $\beta = f'(c_1(k,m))/\tilde{f}'$ and canceling positive factors, the sign of $\partial_1^2 c_1(k,m)$ is the same as the sign of

$$\left(\frac{-\tilde{f}' \tilde{f}'''}{\tilde{f}''} + \frac{f'(c_1(k,m)) f'''(c_1(k,m))}{f''(c_1(k,m))} \right).$$

Let A be the absolute risk aversion coefficient of the function f. The expression in the parenthesis can be written as

$$\frac{A'(m-k-c_1(k,m))}{A^2(m-k-c_1(k,m))} - \frac{A'(c_1(k,m))}{A^2(c_1(k,m))}.$$

Finally, we observe that $m-k-c_1(k,m) < c_1(k,m)$ and use the fact that A is decreasing and convex to conclude that the previous expression is positive. This implies that $\partial_1^2 c_1(k,m) > 0$. □

Lemma 7. *If Assumption 1 holds and $h \in \mathcal{C}$, then*

$$\partial_2 U(s_g^\star, m_g^\star; h \circ g) > 0.$$

Proof. First, we define $k_-^\star := k_-(s_g^\star, m_g^\star)$. Now, let

$$I_1 := \int_{k_-^\star}^{\infty} h'(g(\phi(m_g^\star, c_1(k,m_g^\star),k))) \frac{g'(\phi(m_g^\star, c_1(k,m_g^\star),k))}{f'(\phi(m_g^\star, c_1(k,m_g^\star),k))} \times (-f'(M-m_g^\star) + \alpha(k,m_g^\star) f'(c_1(k,m_g^\star))) \mathrm{d}P(k)$$

and

$$I_2 := (-f'(M-m_g^\star) + \beta f'(m_g^\star - s_g^\star)) \int_{-\infty}^{k_-^\star} h'(g(\phi(m_g^\star, m_g^\star - s_g^\star, s_g^\star - k))) \frac{g'(\phi(m_g^\star, m_g^\star - s_g^\star, k))}{f'(\phi(m_g^\star, m_g^\star - s_g^\star, k))} \mathrm{d}P(k).$$

Notice that $\partial_2 U(s_g^\star, m_g^\star; h \circ g) = I_1 + I_2$. At the same time, Equation 1 and Equation 2 added and multiplied by $h'(g(\phi(m_g^\star, m_g^\star - s_g^\star, s_g^\star - k_-^\star)))$ give

$$\int_{k_-^\star}^{\infty} h'(g(\phi(m_g^\star, m_g^\star - s_g^\star, s_g^\star - k_-^\star))) \frac{g'(\phi(m_g^\star, c_1(k,m_g^\star),k))}{f'(\phi(m_g^\star, c_1(k,m_g^\star),k))} \times (-f'(M-m_g^\star) + \alpha(k,m_g^\star) f'(c_1(k,m_g^\star))) \mathrm{d}P(k)$$
$$+ (-f'(M-m_g^\star) + \beta f'(m_g^\star - s_g^\star)) \int_{-\infty}^{k_-^\star} h'(g(\phi(m_g^\star, m_g^\star - s_g^\star, s_g^\star - k_-^\star))) \frac{g'(\phi(m_g^\star, m_g^\star - s_g^\star, k))}{f'(\phi(m_g^\star, m_g^\star - s_g^\star, k))} \mathrm{d}P(k) = 0.$$

Call the first integral A and the second one B. In order to show that $I_1 + I_2 > 0$, we show that $I_1 > A$ and $I_2 > B$.

We start from I_2. Because h is concave, whenever $k < k_-^\star$ we have

$$h'(g(\phi(m_g^\star, m_g^\star - s_g^\star, s_g^\star - k))) < h'(g(\phi(m_g^\star, m_g^\star - s_g^\star, s_g^\star - k_-^\star))).$$

Combining this and Lemma 4 we conclude that $I_2 > B$.

Now we show the analogous for I_1. By Lemma 5, $-1 < c_1' < 0$, thus

$$\frac{\mathrm{d}}{\mathrm{d}k}(f(c_1(k,m))+f(m-k-c_1(k,m))) = f'(c_1(k,m))c_1'(k)-f'(m-k-c_1(k,m))(1+c_1'(k)) < 0.$$

Since $c_1(k_-^\star, m_g^\star) = m_g^\star - s_g^\star$ and $c_1' < 0$ we have

$$h'(g(\phi(m_g^\star, c_1(k, m_g^\star), k))) > h'(g(\phi(m_g^\star, m_g^\star - s_g^\star, s_g^\star - k_-^\star)))$$

whenever $k > k_-^\star$. Combining Lemma 6 with Lemma 5 we conclude that $\alpha(\cdot, m_g^\star)f'(c_1(\cdot))$ is increasing. Now, for each $k \in \mathbb{R}$, define $t(k) := \frac{h'(g(\phi(m_g^\star, c_1(k,m_g^\star),k)))}{h'(g(\phi(m_g^\star, m_g^\star - s_g^\star, s_g^\star - k_-^\star)))}$ which is smaller than 1, $v(k) := -f'(M - m_g^\star) + \alpha(k, m_g^\star)f'(c_1(k, m_g^\star))$, increasing in k, and $u(k) := \frac{g'(\phi(m_g^\star, m_g^\star - s_g^\star, k))}{f'(\phi(m_g^\star, m_g^\star - s_g^\star, k))} > 0$. Since $B < 0$ (from Lemma 4), we must have $A > 0$. Thus we can apply Lemma 1 to conclude that $I_1 > A$ and the proof is complete. □

For Lemma 8 and the proof of Proposition 2, whenever $(m_g^\star, s_g^\star)$ is well-defined, for each $g, h \in \mathcal{C}$ and $\lambda \in [0,1]$, we redefine

$$Y_s(\lambda; g, h) := \partial_1 U(s_g^\star, m_g^\star; \lambda(h \circ g) + (1-\lambda)g)$$

and we define

$$Y_m(\lambda; g, h) := \partial_2 U(s_g^\star, m_g^\star; \lambda(h \circ g) + (1-\lambda)g).$$

We also define Y_s' and Y_m' as the derivatives with respect to the first argument of Y_s and Y_m, respectively.

Lemma 8. *If Assumption 1 holds, $h \in \mathcal{C}$ is strictly concave, and $(s_g^\star, m_g^\star)$ is unique and interior, then $Y_s'(0; \lambda(h \circ g) + (1-\lambda)g, h) > 0$ and $Y_m'(0; \lambda(h \circ g) + (1-\lambda)g, h) > 0$.*

Proof. The proof follows exactly the same lines of Lemma 3, applying Lemmata 2 and 7. □

Proof of Proposition 2

Proof. Let $\widehat{g}_\lambda := \lambda(h \circ g) + (1-\lambda)g$, $\widehat{s}(\lambda) := s^\star_{\widehat{g}_\lambda}(m)$, and $\widehat{m}(\lambda) := m^\star_{\widehat{g}_\lambda}(m)$. Since $(\widehat{s}(\lambda), \widehat{m}(\lambda))$ is a local (actually global) maximum of $U(\cdot; \widehat{g}_\lambda)$ and $U(\cdot; \widehat{g}_\lambda)$ is differentiable at $(\widehat{s}(\lambda), \widehat{m}(\lambda))$, the Hessian of $U(\widehat{s}(\lambda), \widehat{m}(\lambda); \widehat{g}_\lambda)$, given by

$$H(\lambda) := \begin{bmatrix} \partial_1^2 U(\widehat{s}(\lambda), \widehat{m}(\lambda); \widehat{g}_\lambda) & \partial_1\partial_2 U(\widehat{s}(\lambda), \widehat{m}(\lambda); \widehat{g}_\lambda) \\ \partial_1\partial_2 U(\widehat{s}(\lambda), \widehat{m}(\lambda); \widehat{g}_\lambda) & \partial_2^2 U(\widehat{s}(\lambda), \widehat{m}(\lambda); \widehat{g}_\lambda) \end{bmatrix},$$

is negative defined. This implies that the determinant of this Hessian is non-null and we can apply the IFT. In addition, because $H(\lambda) < 0$, we have $H(\lambda)^{-1} < 0$ and $-H(\lambda)^{-1} > 0$. Let $a(\lambda), b(\lambda)$, and $c(\lambda)$ respect

$$-H(\lambda)^{-1} = \begin{bmatrix} a(\lambda) & b(\lambda) \\ b(\lambda) & c(\lambda) \end{bmatrix},$$

then the IFT tells us that

$$\begin{bmatrix} \widehat{s}'(\lambda) \\ \widehat{m}'(\lambda) \end{bmatrix} = -H(\lambda)^{-1} \begin{bmatrix} Y'_s(0; \widehat{g}_\lambda, h) \\ Y'_m(0; \widehat{g}_\lambda, h) \end{bmatrix} = \begin{bmatrix} a(\lambda)Y'_s(0; \widehat{g}_\lambda, h) + b(\lambda)Y'_m(0; \widehat{g}_\lambda, h) \\ b(\lambda)Y'_s(0; \widehat{g}_\lambda, h) + c(\lambda)Y'_m(0; \widehat{g}_\lambda, h) \end{bmatrix}.$$

Now we show that $a(\lambda), b(\lambda), c(\lambda)$ are positive. Recall that

$$H(\lambda) = \frac{1}{a(\lambda)c(\lambda) - b(\lambda)^2} \begin{bmatrix} -c(\lambda) & b(\lambda) \\ b(\lambda) & -a(\lambda) \end{bmatrix}.$$

Since $-H(\lambda)^{-1}$ is positive defined, $a(\lambda)c(\lambda) - b(\lambda)^2 > 0$. Thus, the sign of $-c(\lambda)$ is the same as $\partial_1^2 U(\widehat{s}(\lambda), \widehat{m}(\lambda); \widehat{g}_\lambda)$, so $c(\lambda) > 0$; the sign of $-a(\lambda)$ is the same as $\partial_2^2 U(\widehat{s}(\lambda), \widehat{m}(\lambda); \widehat{g}_\lambda)$, so $a(\lambda) > 0$; and the sign of $b(\lambda)$ is the same as $\partial_1\partial_2 U(\widehat{s}(\lambda), \widehat{m}(\lambda); \widehat{g}_\lambda)$, so under the assumption that the cross derivative is non-negative, $b(\lambda) \geq 0$.

Because Lemma 8 ensures that $Y'_s(0; \widehat{g}_\lambda, h)$ and $Y'_m(0; \widehat{g}_\lambda, h)$ are positive, we have that $\widehat{s}'(\lambda) > 0$ and $\widehat{m}'(\lambda) > 0$. Applying the Fundamental Theorem of Calculus leads us to the desired result.

□